 Name KEY

Other Trig Formulas

The addition and subtraction formulas are a useful set of identities that are helpful for finding exact value solutions and verifying other identities.

$\sin(a+b) = \sin a \cos b + \cos a \sin b$	$\sin(a-b) = \sin a \cos b - \cos a \sin b$
$\cos(a+b) = \cos a \cos b - \sin a \sin b$	$\cos(a-b) = \cos a \cos b + \sin a \sin b$
$\tan(a+b) = \frac{\tan a + \tan b}{1 - \tan a \tan b}$	$\tan(a-b) = \frac{\tan a - \tan b}{1 + \tan a \tan b}$

Example:

Find the exact value of each expression.

a) $\cos 75°$

b) $\cos \frac{\pi}{12}$

Solution:

a) Find two familiar angles that sum up to 75º.

$30°$ & $45°$ *work*! *So*, $\cos 75° = \cos(30° + 45°) = \cos 30° \cdot \cos 45° - \sin 30° \cdot \sin 45°$ by making $a = 30°$ & $b = 45°$

Find exact values (from the unit circle) and substitute into the equation!

$\cos 30° \cdot \cos 45° - \sin 30° \cdot \sin 45° = \frac{\sqrt{3}}{2} \cdot \frac{\sqrt{2}}{2} - \frac{1}{2} \cdot \frac{\sqrt{2}}{2} = \frac{\sqrt{6}}{4} - \frac{\sqrt{2}}{4} = \frac{\sqrt{6} - \sqrt{2}}{4}$

This would have worked the same if we used $a = 45°$ & $b = 30°$ because addition is commutative.

b) Find two familiar angles that have a difference of $\frac{\pi}{12}$.

This one is a little more difficult because it's in radians and we're looking for the *difference*. So, convert all of the unit circles radian measures into twelfths. We find that many different measure will work once we do this. Here is one example: $\frac{\pi}{4} - \frac{\pi}{6} = \frac{3\pi}{12} - \frac{2\pi}{12}$, *so* $\cos \frac{\pi}{12} = \cos \frac{\pi}{4} - \frac{\pi}{6} = \cos \frac{\pi}{4} \cdot \cos \frac{\pi}{6} + \sin \frac{\pi}{4} \cdot \sin \frac{\pi}{6}$ by making $a = \frac{\pi}{4}$ & $b = \frac{\pi}{6}$

Find exact values (from the unit circle) and substitute into the equation!

$\cos \frac{\pi}{4} \cdot \cos \frac{\pi}{6} + \sin \frac{\pi}{4} \cdot \sin \frac{\pi}{6} = \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{3}}{2} + \frac{\sqrt{2}}{2} \cdot \frac{1}{2} = \frac{\sqrt{6}}{4} + \frac{\sqrt{2}}{4} = \frac{\sqrt{6}+\sqrt{2}}{4}$

For #'s 1 - 6, use an addition or subtraction formula to find the exact value of the expression, as demonstrated above.

1. $\sin 15° = \sin(45-30)$

$= \sin 45 \cdot \cos 30 - \cos 45 \cdot \sin 30$

$= \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{3}}{2} - \frac{\sqrt{2}}{2} \cdot \frac{1}{2} = \boxed{\frac{\sqrt{6}-\sqrt{2}}{4}}$

2. $\cos 165° = \cos(135+30)$

$= \cos 135 \cos 30 - \sin 135 \sin 30$

$= -\frac{\sqrt{2}}{2} \cdot \frac{\sqrt{3}}{2} - \frac{\sqrt{2}}{2} \cdot \frac{1}{2}$

$\boxed{= \frac{-\sqrt{6}-\sqrt{2}}{4}}$

3. $\tan 105° = \tan(60+45)$

$= \dfrac{\tan 60 + \tan 45}{1 - \tan 60 \tan 45} = \dfrac{\sqrt{3}+1}{1-\sqrt{3}\cdot 1} = \dfrac{(\sqrt{3}+1)}{1-\sqrt{3}}\cdot\dfrac{(1+\sqrt{3})}{(1+\sqrt{3})}$

$= \dfrac{3+1+2\sqrt{3}}{1-3}$

$= \dfrac{4+2\sqrt{3}}{-2}$

$= \dfrac{2(2+\sqrt{3})}{-2} = \boxed{-(2+\sqrt{3})}$

4. $\sin\frac{\pi}{12} = \sin\left(\frac{\pi}{3}-\frac{\pi}{4}\right) =$

$\sin\frac{\pi}{3}\cos\frac{\pi}{4} - \cos\frac{\pi}{3}\sin\frac{\pi}{4} =$

$\frac{\sqrt{3}}{2}\cdot\frac{\sqrt{2}}{2} - \frac{1}{2}\cdot\frac{\sqrt{2}}{2} = \boxed{\dfrac{\sqrt{6}-\sqrt{2}}{4}}$

5. $\sin\frac{11\pi}{12} = \sin\left(\frac{3\pi}{4}+\frac{\pi}{6}\right) =$

$\sin\frac{3\pi}{4}\cos\frac{\pi}{6} + \cos\frac{3\pi}{4}\sin\frac{\pi}{6} =$

$\frac{\sqrt{2}}{2}\cdot\frac{\sqrt{3}}{2} + -\frac{\sqrt{2}}{2}\cdot\frac{1}{2} = \boxed{\dfrac{\sqrt{6}-\sqrt{2}}{4}}$

6. $\sin 75° = \sin(30+45)$

$= \sin 30 \cos 45 + \cos 30 \sin 45$

$= \frac{1}{2}\cdot\frac{\sqrt{2}}{2} + \frac{\sqrt{3}}{2}\cdot\frac{\sqrt{2}}{2}$

$\boxed{= \dfrac{\sqrt{2}+\sqrt{6}}{4}}$

For #'s 7 – 10, use an addition or subtraction formula to write the expression as a trigonometric function of one number and find its exact value.

Match up formula!

7. $\sin 18° \cos 27° + \cos 18° \sin 27°$

$\sin(18+27) = \boxed{\dfrac{\sqrt{2}}{2}}$

$\sin 45 =$

8. $\cos\frac{3\pi}{7}\cos\frac{2\pi}{21} + \sin\frac{3\pi}{7}\sin\frac{2\pi}{21}$

$\cos\left(\frac{3\pi}{7}-\frac{2\pi}{21}\right) = \cos\frac{\pi}{3} = \boxed{\dfrac{1}{2}}$

9. $\dfrac{\tan 73° - \tan 13°}{1+\tan 73° \tan 13°}$ $= \tan(73-13)$

$= \tan 60$

$\boxed{= \sqrt{3}}$

10. $\cos\frac{13\pi}{15}\cos\left(-\frac{\pi}{5}\right) - \sin\frac{13\pi}{15}\sin\left(-\frac{\pi}{5}\right)$

$\cos\left(\frac{13\pi}{15}+\frac{-\pi}{5}\right) = \cos\frac{2\pi}{3} = \boxed{-\dfrac{1}{2}}$

Use the addition formula above to prove:

a) $\sin 2\theta = 2\sin\theta\cos\theta$

$\sin(\theta+\theta) = \sin\theta\cos\theta + \cos\theta\sin\theta = 2\sin\theta\cos\theta$

$\boxed{2\sin\theta\cos\theta = 2\sin\theta\cos\theta}$

b) $\cos 2\theta = \cos^2\theta - \sin^2\theta$

$\cos(\theta+\theta) = \cos^2\theta - \sin^2\theta$

$\cos\theta\cos\theta - \sin\theta\sin\theta = \cos^2\theta - \sin^2\theta$

$\boxed{\cos^2\theta - \sin^2\theta = \cos^2\theta - \sin^2\theta}$

The two above formulas are the double – angle formulas, used to find the values of the trigonometric functions that are 2θ from their values at θ. There also exists one for tangent, as seen in the following table.

$$\sin 2\theta = 2\sin\theta\cos\theta$$

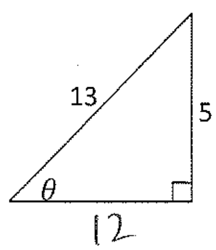
$$\cos 2\theta = \cos^2\theta - \sin^2\theta = 1 - 2\sin^2\theta = 2\cos^2\theta - 1$$

$$\tan 2\theta = \frac{2\tan\theta}{1 - \tan^2\theta}$$

For #'s 11 – 14, find $\sin 2\theta, \cos 2\theta,$ *and* $\tan 2\theta$ from the given information.

11. $\sin\theta = \frac{5}{13}$ (x in quadrant I)

Example: So this means that $\sin 2\theta = 2\sin\theta\cos\theta = 2\cdot\frac{5}{13}\cdot\cos\theta$. To finish this and calculate the value of $\sin 2\theta$, we need to find the ratio for $\cos\theta$. We can do this by looking at a right triangle and finding the missing side. Using the Pythagorean Theorem on the right triangle below gives us an adjacent side of 12.



Therefore, $\sin 2\theta = 2\sin\theta\cos\theta = 2\cdot\frac{5}{13}\cdot\frac{12}{13} = \mathbf{\frac{120}{169}}$. The quadrant means that our values will be positive for our two leg lengths.

Now find $\cos 2\theta$ & $\tan 2\theta$ using the same right triangle above.

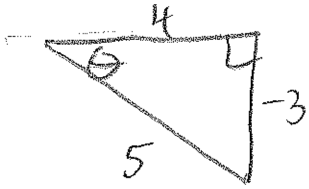
$$\cos 2\theta = \cos^2\theta - \sin^2\theta = \left(\frac{12}{13}\right)^2 - \left(\frac{5}{13}\right)^2 = \frac{144}{169} - \frac{25}{169} = \boxed{\frac{119}{169}}$$

$$\tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta} = \frac{2\left(\frac{5}{12}\right)}{1-\left(\frac{5}{12}\right)^2} = \frac{\frac{10}{12}}{1-\frac{25}{144}} = \frac{10}{12}\cdot\frac{144}{119} = \boxed{\frac{120}{119}}$$

$$\text{Also } \tan 2\theta = \frac{\sin 2\theta}{\cos 2\theta} = \frac{\frac{120}{169}}{\frac{119}{169}} = \boxed{\frac{120}{119}}$$

12. $\cos\theta = \frac{4}{5}$ $(\csc\theta < 0)$ $\frac{1}{y} < 0$ / so QIII or QIV

We use QIV because $\cos\theta = \frac{4}{5}$ (positive)



$\sin 2\theta = 2\sin\theta\cos\theta$

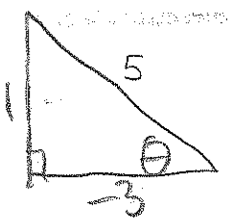
$= 2\left(\frac{-3}{5}\right)\left(\frac{4}{5}\right) = \boxed{\frac{-24}{25}}$

$\cos 2\theta = \cos^2\theta - \sin^2\theta = \left(\frac{4}{5}\right)^2 - \left(\frac{-3}{5}\right)^2 = \frac{16}{25} - \frac{9}{25} = \boxed{\frac{7}{25}}$

$\tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta} = \frac{2\left(\frac{-3}{4}\right)}{1-\left(\frac{-3}{4}\right)^2} = \frac{\frac{-6}{4}}{\frac{7}{16}} = \frac{-6}{4}\cdot\frac{16}{7} = \boxed{\frac{-24}{7}}$

13. $\tan\theta = \frac{-4}{3}$ (θ in quadrant II)

really means adj is negative



$\sin 2\theta = 2\sin\theta\cos\theta$

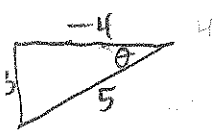
$= 2\left(\frac{4}{5}\right)\left(\frac{-3}{5}\right) = \boxed{\frac{-24}{25}}$

$\cos 2\theta = \cos^2\theta - \sin^2\theta$

$= \left(\frac{-3}{5}\right)^2 - \left(\frac{4}{5}\right)^2 = \frac{9}{25} - \frac{16}{25} = \boxed{\frac{-7}{25}}$

$\tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta} = \frac{2\left(\frac{-4}{3}\right)}{1-\left(\frac{-4}{3}\right)^2} = \frac{\frac{-8}{3}}{\frac{-7}{9}} = \frac{8}{3}\cdot\frac{9}{7} = \boxed{\frac{24}{7}}$

14. $\sin\theta = \frac{-3}{5}$ (θ in quadrant III)



$\sin 2\theta = 2\sin\theta\cos\theta$

$= 2\left(\frac{-3}{5}\right)\left(\frac{-4}{5}\right) = \boxed{\frac{24}{25}}$

$\cos 2\theta = \cos^2\theta - \sin^2\theta$

$= \left(\frac{-4}{5}\right)^2 - \left(\frac{-3}{5}\right)^2 = \frac{16}{25} - \frac{9}{25} = \boxed{\frac{7}{25}}$

$\tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta} = \frac{2\left(\frac{-3}{-4}\right)}{1-\left(\frac{-3}{-4}\right)^2} = \frac{\frac{6}{4}}{\frac{7}{16}} = \frac{6}{4}\cdot\frac{16}{7}$

$= \boxed{\frac{24}{7}}$