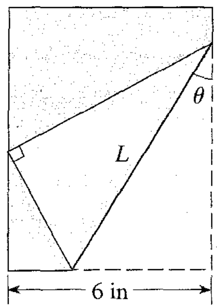
73. The lower right-hand corner of a long piece of paper 6 in. wide is folded over to the left-hand edge as shown. The length L of the fold depends on the angle θ. Show that

$$L = \frac{3}{\sin\theta\cos^2\theta}$$



74. Let $3x = \pi/3$ and let $y = \cos x$. Use the result of Example 2 to show that y satisfies the equation

$$8y^3 - 6y - 1 = 0$$

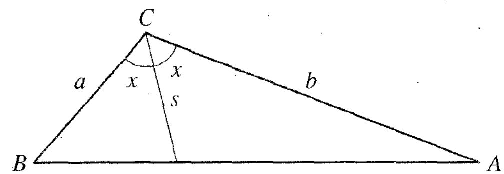
NOTE: This equation has roots of a certain kind that are used in showing that the angle $\pi/3$ cannot be trisected using a ruler and compass only.

75. (a) Show that there is a polynomial $P(t)$ of degree 4 such that $\cos 4x = P(\cos x)$ (see Example 2).
(b) Show that there is a polynomial $Q(t)$ of degree 5 such that $\cos 5x = Q(\cos x)$.

NOTE: In general, there is a polynomial $P_n(t)$ of degree n such that $\cos nx = P_n(\cos x)$. These polynomials are called Tchebycheff polynomials, after the Russian mathematician P. L. Tchebycheff (1821–1894).

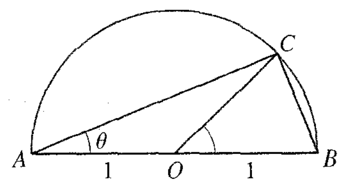
76. In triangle ABC (see the figure) the line segment s bisects angle C. Show that the length of s is given by

$$s = \frac{2ab\cos x}{a + b}$$



DISCOVERY • DISCUSSION

77. Geometric Proof of a Double-Angle Formula Use the figure to prove that $\sin 2\theta = 2\sin\theta\cos\theta$.



Hint: Find the area of triangle ABC in two different ways. You will need the following facts from geometry:

An angle inscribed in a semicircle is a right angle, so $\angle ACB$ is a right angle.

The central angle subtended by the chord of a circle is twice the angle subtended by the chord on the circle, so $\angle BOC$ is 2θ.

7.4 INVERSE TRIGONOMETRIC FUNCTIONS

If f is a one-to-one function with domain A and range B, then its inverse f^{-1} is the function with domain B and range A defined by

$$f^{-1}(x) = y \iff f(y) = x$$

(See Section 2.7.) In other words, f^{-1} is the rule that reverses the action of f. Figure 1 represents the actions of f and f^{-1} graphically.

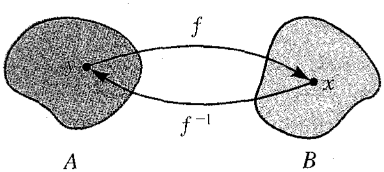


FIGURE 1
$f^{-1}(x) = y \iff f(y) = x$

For a function to have an inverse, it must be one-to-one. Since the trigonometric functions are not one-to-one, they do not have inverses. It is possible, however, to restrict the domains of the trigonometric functions in such a way that the resulting functions are one-to-one.

THE INVERSE SINE FUNCTION

Let's first consider the sine function. There are many ways to restrict the domain of sine so that the new function is one-to-one. A natural way to do this is to restrict the domain to the interval $[-\pi/2, \pi/2]$. The reason for this choice is that sine attains each of its values exactly once on this interval. We write $\operatorname{Sin} x$ (with a capital S) for the new function, which has the domain $[-\pi/2, \pi/2]$ and the same values as $\sin x$ on this interval. The graphs of $\sin x$ and $\operatorname{Sin} x$ are shown in Figure 2. The function $\operatorname{Sin} x$ is one-to-one (by the Horizontal Line Test), and so has an inverse.

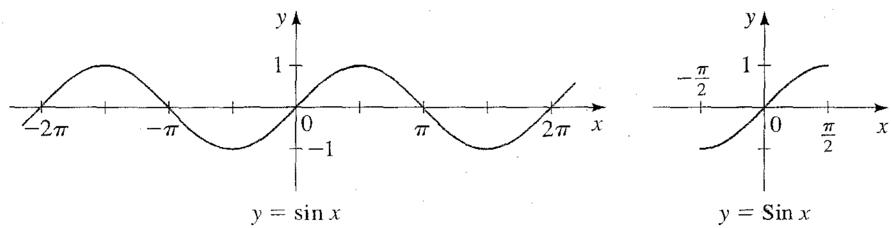


FIGURE 2

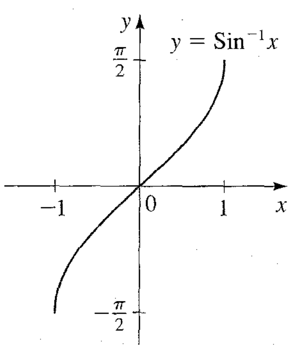


FIGURE 3

The inverse of the function $\operatorname{Sin} x$ is the function $\operatorname{Sin}^{-1} x$ defined by

$$\operatorname{Sin}^{-1} x = y \iff \operatorname{Sin} y = x$$

for $-1 \leqslant x \leqslant 1$ and $-\pi/2 \leqslant y \leqslant \pi/2$. The graph of $y = \operatorname{Sin}^{-1} x$ is shown in Figure 3; it is obtained by reflecting the graph of $y = \operatorname{Sin} x$ in the line $y = x$. It is customary to write $\operatorname{Sin}^{-1} x$ simply as $\sin^{-1} x$.

DEFINITION OF THE INVERSE SINE FUNCTION

The **inverse sine function** is the function $\mathbf{sin^{-1}}$ with domain $[-1, 1]$ and range $[-\pi/2, \pi/2]$ defined by

$$\sin^{-1} x = y \iff \sin y = x$$

The inverse sine function is also called **arcsine** and is denoted by **arcsin**.

Thus, $\sin^{-1} x$ *is the number in the interval* $[-\pi/2, \pi/2]$ *whose sine is* x. In other words, $\sin(\sin^{-1} x) = x$. In fact, from the general properties of inverse functions studied in Section 2.7, we have the following relations.

$$\sin(\sin^{-1} x) = x \qquad \text{for } -1 \leqslant x \leqslant 1$$

$$\sin^{-1}(\sin x) = x \qquad \text{for } -\frac{\pi}{2} \leqslant x \leqslant \frac{\pi}{2}$$

EXAMPLE 1 ■ Evaluating the Inverse Sine Function

Find (a) $\sin^{-1}\frac{1}{2}$, (b) $\sin^{-1}(-\frac{1}{2})$, and (c) $\sin^{-1}\frac{3}{2}$.

SOLUTION

(a) The number in the interval $[-\pi/2, \pi/2]$ whose sine is $\frac{1}{2}$ is $\pi/6$. Thus, $\sin^{-1}\frac{1}{2} = \pi/6$.

(b) Again, $\sin^{-1}(-\frac{1}{2})$ is the number in the interval $[-\pi/2, \pi/2]$ whose sine is $-\frac{1}{2}$. Since $\sin(-\pi/6) = -\frac{1}{2}$, we have $\sin^{-1}(-\frac{1}{2}) = -\pi/6$.

(c) Since $\frac{3}{2} > 1$, it is not in the domain of $\sin^{-1}x$, so $\sin^{-1}\frac{3}{2}$ is not defined. ■

EXAMPLE 2 ■ Using a Calculator to Evaluate Inverse Sine

Find approximate values for (a) $\sin^{-1}(0.82)$ and (b) $\sin^{-1}\frac{1}{3}$.

SOLUTION Since no rational multiple of π has a sine of 0.82 or $\frac{1}{3}$, we use a calculator to approximate these values. Using the [INV] [SIN], or [SIN⁻¹], or [ARCSIN] key(s) on the calculator (with the calculator in radian mode), we get

$$\text{(a)}\ \sin^{-1}(0.82) \approx 0.96141 \qquad \text{(b)}\ \sin^{-1}\tfrac{1}{3} \approx 0.33984$$ ■

EXAMPLE 3 ■ Composing Trigonometric Functions and Their Inverses

Find $\cos(\sin^{-1}\frac{3}{5})$.

SOLUTION 1 It is easy to find $\sin(\sin^{-1}\frac{3}{5})$. In fact, by the properties of inverse functions, this value is exactly $\frac{3}{5}$. To find $\cos(\sin^{-1}\frac{3}{5})$, we reduce this to the easier problem by writing the cosine function in terms of the sine function. Let $u = \sin^{-1}\frac{3}{5}$. Since $-\pi/2 \leqslant u \leqslant \pi/2$, $\cos u$ is positive and we can write

$$\cos u = +\sqrt{1 - \sin^2 u}$$

Thus

$$\cos(\sin^{-1}\tfrac{3}{5}) = \sqrt{1 - \sin^2(\sin^{-1}\tfrac{3}{5})}$$

$$= \sqrt{1 - (\tfrac{3}{5})^2} = \sqrt{1 - \tfrac{9}{25}} = \sqrt{\tfrac{16}{25}} = \tfrac{4}{5}$$

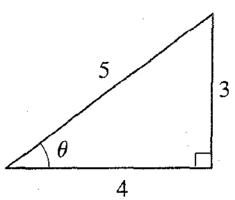


FIGURE 4

SOLUTION 2 Let $\theta = \sin^{-1}\frac{3}{5}$. Then θ is the number in the interval $[-\pi/2, \pi/2]$ whose sine is $\frac{3}{5}$. Let's interpret θ as an angle and draw a right triangle with θ as one of its acute angles, with opposite side 3 and hypotenuse 5 (see Figure 4). The remaining leg of the triangle is found by the Pythagorean Theorem to be 4. From the figure we get

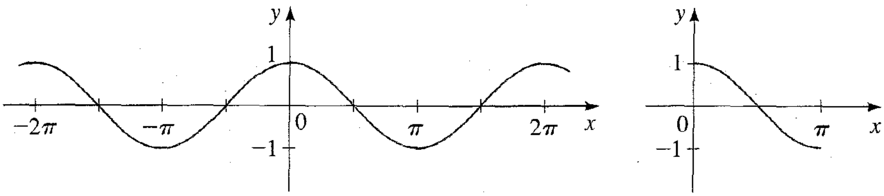
$$\cos(\sin^{-1}\tfrac{3}{5}) = \cos\theta = \tfrac{4}{5}$$ ■

From Solution 2 of Example 3 we can immediately find the values of the other trigonometric functions of $\theta = \sin^{-1}\frac{3}{5}$ from the triangle. Thus,

$$\tan(\sin^{-1}\tfrac{3}{5}) = \tfrac{3}{4} \qquad \sec(\sin^{-1}\tfrac{3}{5}) = \tfrac{5}{4} \qquad \csc(\sin^{-1}\tfrac{3}{5}) = \tfrac{5}{3}$$

THE INVERSE COSINE FUNCTION

If the domain of the cosine function is restricted to the interval $[0, \pi]$, the resulting function is one-to-one and so has an inverse. We choose this interval because on it, cosine attains each of its values exactly once (see Figure 5).



$y = \cos x$ $\qquad$ $y = \cos x,\ 0 \leq x \leq \pi$

FIGURE 5

DEFINITION OF THE INVERSE COSINE FUNCTION

The **inverse cosine function** is the function $\mathbf{cos^{-1}}$ with domain $[-1, 1]$ and range $[0, \pi]$ defined by

$$\cos^{-1} x = y \quad \Leftrightarrow \quad \cos y = x$$

The inverse cosine function is also called **arccosine** and is denoted by **arccos**.

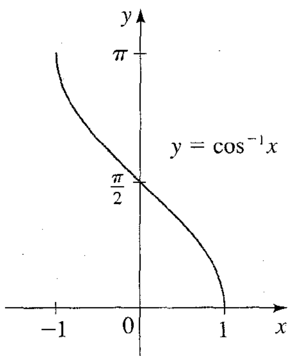


FIGURE 6

Thus, $y = \cos^{-1} x$ *is the number in the interval* $[0, \pi]$ *whose cosine is* x. The following relations follow from the inverse function properties.

$\cos(\cos^{-1} x) = x$	for $-1 \leq x \leq 1$
$\cos^{-1}(\cos x) = x$	for $0 \leq x \leq \pi$

The graph of $y = \cos^{-1} x$ is sketched in Figure 6; it is obtained by reflecting the graph of $y = \cos x$, $0 \leq x \leq \pi$, in the line $y = x$.

EXAMPLE 4 ■ Evaluating the Inverse Cosine Function

Find (a) $\cos^{-1}(\sqrt{3}/2)$, (b) $\cos^{-1} 0$, and (c) $\cos^{-1} \frac{5}{7}$.

SOLUTION

(a) The number in the interval $[0, \pi]$ whose cosine is $\sqrt{3}/2$ is $\pi/6$. Thus, $\cos^{-1}(\sqrt{3}/2) = \pi/6$.

(b) Since $\cos(\pi/2) = 0$, it follows that $\cos^{-1} 0 = \pi/2$.

(c) Since no rational multiple of π has cosine $\frac{5}{7}$, we use a calculator to find this value approximately: $\cos^{-1} \frac{5}{7} \approx 0.77519$. ■

EXAMPLE 5 ■ Composing Trigonometric Functions and Their Inverses

Write $\sin(\cos^{-1}x)$ and $\tan(\cos^{-1}x)$ as algebraic expressions in x for $-1 \leqslant x \leqslant 1$.

SOLUTION 1 Let $u = \cos^{-1}x$. We need to find $\sin u$ and $\tan u$ in terms of x. As in Example 3 the idea here is to write sine and tangent in terms of cosine. We have

$$\sin u = \pm\sqrt{1 - \cos^2 u} \qquad \text{and} \qquad \tan u = \frac{\sin u}{\cos u} = \frac{\pm\sqrt{1 - \cos^2 u}}{\cos u}$$

To choose the proper signs, note that u lies in the interval $[0, \pi]$ because $u = \cos^{-1}x$. Since $\sin u$ is positive on this interval, the + sign is the correct choice. Now substituting $u = \cos^{-1}x$ in the displayed equations and using the relation $\cos(\cos^{-1}x) = x$ gives

$$\sin(\cos^{-1}x) = \sqrt{1 - x^2} \qquad \text{and} \qquad \tan(\cos^{-1}x) = \frac{\sqrt{1 - x^2}}{x}$$

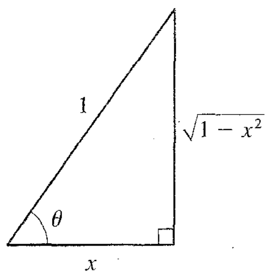


FIGURE 7

$\cos\theta = \frac{x}{1} = x$

SOLUTION 2 Let $\theta = \cos^{-1}x$, so $\cos\theta = x$. In Figure 7 we draw a right triangle with an acute angle θ, adjacent side x, and hypotenuse 1. By the Pythagorean Theorem, the remaining leg is $\sqrt{1 - x^2}$. From the figure,

$$\sin(\cos^{-1}x) = \sin\theta = \sqrt{1 - x^2} \qquad \text{and} \qquad \tan(\cos^{-1}x) = \tan\theta = \frac{\sqrt{1 - x^2}}{x}$$

■

NOTE In Solution 2 of Example 5, it may seem that because we are sketching a triangle, the angle $\theta = \cos^{-1}x$ must be acute. But it turns out that the triangle method works for any θ and for any x. The domains and ranges of all six inverse trigonometric functions have been chosen in such a way that we can always use a triangle to find $S(T^{-1}(x))$, where S and T are any trigonometric functions.

EXAMPLE 6 ■ Composing a Trigonometric Function and an Inverse

Write $\sin(2\cos^{-1}x)$ as an algebraic expression in x for $-1 \leqslant x \leqslant 1$.

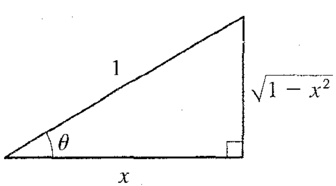


FIGURE 8

$\cos\theta = \frac{x}{1} = x$

SOLUTION Let $\theta = \cos^{-1}x$ and sketch a triangle as shown in Figure 8. We need to find $\sin 2\theta$, but from the triangle we can find the trigonometric function only of θ, not of 2θ. The double-angle identity for sine is useful here. We have

$$\begin{aligned} \sin(2\cos^{-1}x) &= \sin 2\theta \\ &= 2\sin\theta\cos\theta && \text{Double-angle formula} \\ &= 2\sqrt{1 - x^2}\,x && \text{From triangle} \\ &= 2x\sqrt{1 - x^2} \end{aligned}$$

■

THE INVERSE TANGENT FUNCTION

We restrict the domain of the tangent function to the interval $(-\pi/2, \pi/2)$ in order to obtain a one-to-one function.

DEFINITION OF THE INVERSE TANGENT FUNCTION

The **inverse tangent function** is the function $\tan^{-1}$ with domain $\mathbb{R}$ and range $(-\pi/2, \pi/2)$ defined by

$$\tan^{-1}x = y \iff \tan y = x$$

The inverse tangent function is also called **arctangent** and is denoted by **arctan**.

Thus, $\tan^{-1}x$ *is the number in the interval* $(-\pi/2, \pi/2)$ *whose tangent is* x. The following relations follow from the inverse function properties.

$$\tan(\tan^{-1}x) = x \qquad \text{for } x \in \mathbb{R}$$

$$\tan^{-1}(\tan x) = x \qquad \text{for } -\frac{\pi}{2} < x < \frac{\pi}{2}$$

Figure 9 shows the graph of $y = \tan x$ on the interval $(-\pi/2, \pi/2)$ and the graph of its inverse function, $y = \tan^{-1}x$.

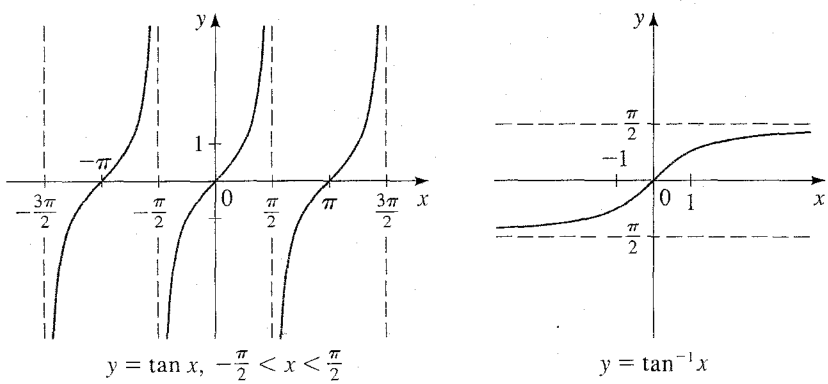


FIGURE 9

EXAMPLE 7 ■ Evaluating the Inverse Tangent Function

Find (a) $\tan^{-1}1$, (b) $\tan^{-1}\sqrt{3}$, and (c) $\tan^{-1}(-20)$.

SOLUTION

(a) The number in the interval $(-\pi/2, \pi/2)$ with tangent 1 is $\pi/4$. Thus, $\tan^{-1}1 = \pi/4$.

(b) Since $\tan(\pi/3) = \sqrt{3}$, we have $\tan^{-1}\sqrt{3} = \pi/3$.

(c) We use a calculator to find that $\tan^{-1}(-20) \approx -1.52084$. ■

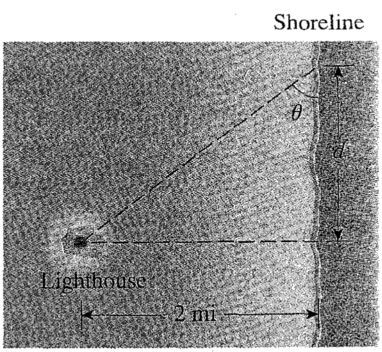


FIGURE 10

EXAMPLE 8 ■ The Angle of a Beam of Light

A lighthouse is located on an island that is 2 mi off a straight shoreline (see Figure 10). Express the angle formed by the beam of light and the shoreline in terms of the distance d in the figure.

SOLUTION From the figure, we see that $\tan\theta = 2/d$. Taking the inverse tangent of both sides, we get

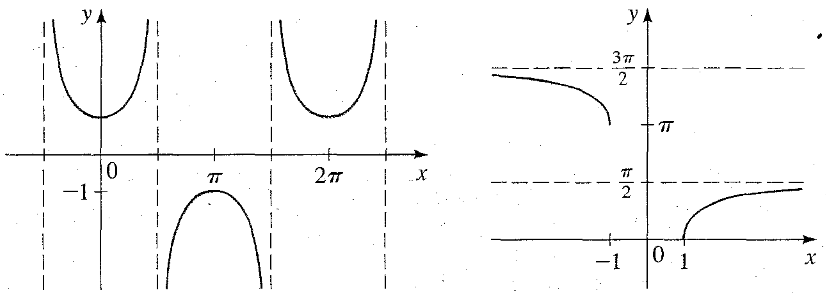
$$\tan^{-1}(\tan\theta) = \tan^{-1}\left(\frac{2}{d}\right)$$

$$\theta = \tan^{-1}\left(\frac{2}{d}\right) \qquad \text{Cancellation property}$$

■

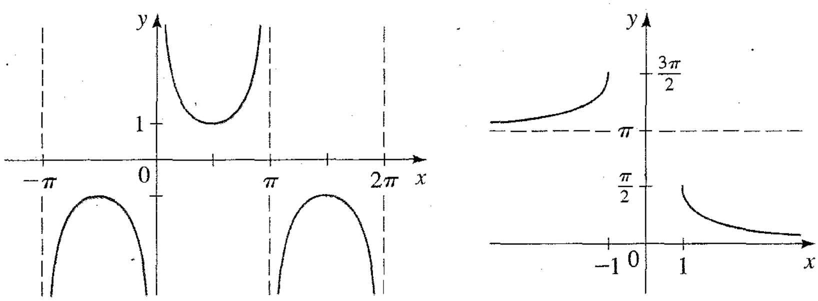
THE INVERSE SECANT, COSECANT, AND COTANGENT FUNCTIONS

To define the inverse functions of the secant, cosecant, and cotangent functions, we restrict the domain of each function to a set on which it is one-to-one and on which it attains all its values. Although any interval satisfying these criteria is appropriate, we choose to restrict the domains in a way that simplifies the choice of sign in computations involving inverse trigonometric functions. The choices we make are also appropriate for calculus. This explains the seemingly strange restriction for the domains of the secant and cosecant functions. We end this section by displaying the graphs of the secant, cosecant, and cotangent functions with their restricted domains and the graphs of their inverse functions (Figures 11–13).



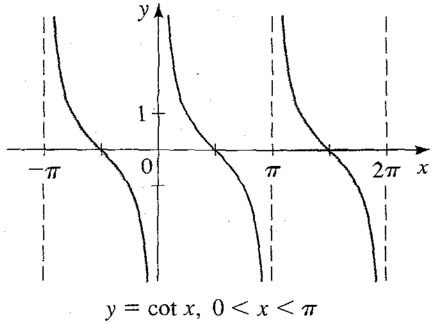
$y = \sec x,\ 0 \leqslant x < \frac{\pi}{2},\ \pi \leqslant x < \frac{3\pi}{2}$ $\qquad y = \sec^{-1} x$

FIGURE 11
The inverse secant function



$y = \csc x,\ 0 < x \leqslant \frac{\pi}{2},\ \pi < x \leqslant \frac{3\pi}{2}$ $\qquad y = \csc^{-1} x$

FIGURE 12
The inverse cosecant function



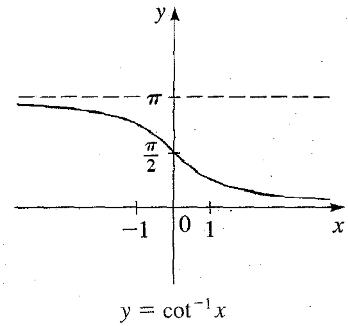


FIGURE 13
The inverse cotangent function

7.4 EXERCISES

1–8 ■ Find the exact value of each expression, if it is defined.

1. (a) $\sin^{-1}\frac{1}{2}$ (b) $\cos^{-1}\frac{1}{2}$ (c) $\cos^{-1}2$

2. (a) $\sin^{-1}\frac{\sqrt{3}}{2}$ (b) $\cos^{-1}\frac{\sqrt{3}}{2}$ (c) $\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right)$

3. (a) $\sin^{-1}\frac{\sqrt{2}}{2}$ (b) $\cos^{-1}\frac{\sqrt{2}}{2}$ (c) $\sin^{-1}\left(-\frac{\sqrt{2}}{2}\right)$

4. (a) $\tan^{-1}\sqrt{3}$ (b) $\tan^{-1}(-\sqrt{3})$ (c) $\sin^{-1}\sqrt{3}$

5. (a) $\sin^{-1}1$ (b) $\cos^{-1}1$ (c) $\cos^{-1}(-1)$

6. (a) $\tan^{-1}1$ (b) $\tan^{-1}(-1)$ (c) $\tan^{-1}0$

7. (a) $\tan^{-1}\frac{\sqrt{3}}{3}$ (b) $\tan^{-1}\left(-\frac{\sqrt{3}}{3}\right)$ (c) $\sin^{-1}(-2)$

8. (a) $\sin^{-1}0$ (b) $\cos^{-1}0$ (c) $\cos^{-1}(-\frac{1}{2})$

9–10 ■ Use a calculator to find an approximate value of each expression correct to five decimal places.

9. (a) $\sin^{-1}(0.7688)$ (b) $\cos^{-1}(-0.5014)$

10. (a) $\cos^{-1}(0.3388)$ (b) $\tan^{-1}(15.2000)$

11–26 ■ Find the exact value of the expression, if it is defined.

11. $\sin(\sin^{-1}\frac{1}{3})$

12. $\cos(\cos^{-1}\frac{3}{4})$

13. $\tan(\tan^{-1}10)$

14. $\sin(\sin^{-1}10)$

15. $\cos^{-1}\left(\cos\frac{\pi}{3}\right)$

16. $\tan^{-1}\left(\tan\frac{\pi}{6}\right)$

17. $\sin^{-1}\left[\sin\left(-\frac{\pi}{6}\right)\right]$

18. $\sin^{-1}\left(\sin\frac{5\pi}{6}\right)$

19. $\tan^{-1}\left(\tan\frac{2\pi}{3}\right)$

20. $\cos^{-1}\left[\cos\left(-\frac{\pi}{4}\right)\right]$

21. $\tan(\sin^{-1}\frac{1}{2})$

22. $\sin(\sin^{-1}0)$

23. $\cos\left(\sin^{-1}\frac{\sqrt{3}}{2}\right)$

24. $\tan\left(\sin^{-1}\frac{\sqrt{2}}{2}\right)$

25. $\tan^{-1}\left(2\sin\frac{\pi}{3}\right)$

26. $\cos^{-1}\left(\sqrt{3}\sin\frac{\pi}{6}\right)$

27–38 ■ Evaluate the expression by sketching a triangle, as in Solution 2 of Example 3.

27. $\sin(\cos^{-1}\frac{3}{5})$

28. $\tan(\sin^{-1}\frac{4}{5})$

29. $\sin(\tan^{-1}\frac{12}{5})$

30. $\cos(\tan^{-1}5)$

31. $\sec(\sin^{-1}\frac{12}{13})$

32. $\csc(\cos^{-1}\frac{7}{25})$

33. $\cos(\tan^{-1}2)$

34. $\cot(\sin^{-1}\frac{2}{3})$

35. $\sin(2\cos^{-1}\frac{3}{5})$

36. $\tan(2\tan^{-1}\frac{5}{13})$

37. $\sin(\sin^{-1}\frac{1}{2} + \cos^{-1}\frac{1}{2})$

38. $\cos(\sin^{-1}\frac{3}{5} - \cos^{-1}\frac{3}{5})$

39–46 ■ Rewrite the expression as an algebraic expression in x.

39. $\cos(\sin^{-1}x)$

40. $\sin(\tan^{-1}x)$

41. $\tan(\sin^{-1}x)$

42. $\cos(\tan^{-1}x)$

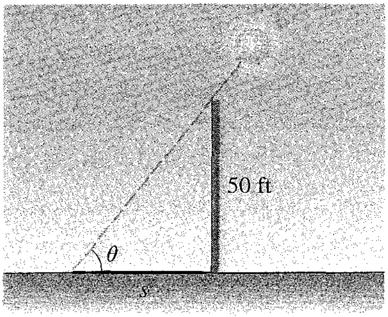
43. $\cos(2\tan^{-1}x)$

44. $\sin(2\sin^{-1}x)$

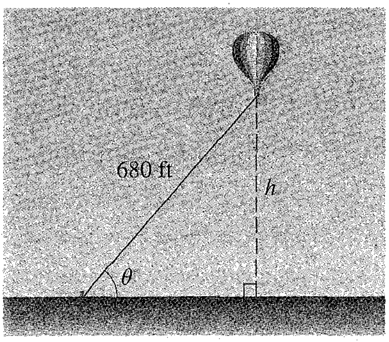
45. $\cos(\cos^{-1}x + \sin^{-1}x)$

46. $\sin(\tan^{-1}x - \sin^{-1}x)$

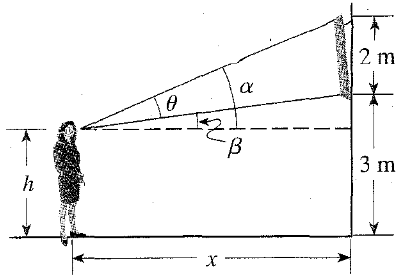
47. A 50-ft pole casts a shadow of length s as shown. Express the angle θ of elevation of the sun in terms of the length s of the shadow.



48. A 680-ft rope anchors a hot-air balloon. Express the angle θ in the figure as a function of the height h.



49. A painting that is 2 m high hangs in a museum with its bottom edge 3 m above the floor. A person whose eye level is h meters above the floor stands at a distance of x meters directly in front of the painting. The size of the painting as it appears to the viewer is determined by the size of the angle θ that the painting subtends at the viewer's eyes (see the figure). The larger θ is, the larger the apparent size of the painting. The angle θ depends on the distance x; in other words, the angle θ is a function of x.



(a) Show that

$$\theta = \tan^{-1}\left(\frac{2x}{x^2 + (3 - h)(5 - h)}\right)$$

[*Hint:* Use the subtraction formula for tangent and the fact that $\theta = \alpha - \beta$.]

(b) Assume the viewer's eye level is $h = 2$ m. Find the angle θ (in degrees) subtended by the picture for $x = 0.5$, 2, and 5 m.

50–51 ■ (a) Graph the function and make a conjecture, and (b) prove that your conjecture is true.

50. $y = \tan^{-1}x + \tan^{-1}\dfrac{1}{x}$

51. $y = \sin^{-1}x + \cos^{-1}x$

52–53 ■ (a) Use a graphing device to find all solutions to the equation, correct to two decimal places, and (b) find the exact solution.

52. $\sin^{-1}x - \cos^{-1}x = 0$

53. $\tan^{-1}x + \tan^{-1}2x = \dfrac{\pi}{4}$

DISCOVERY • DISCUSSION

54. Two Different Compositions The functions

$$f(x) = \sin(\sin^{-1}x) \qquad \text{and} \qquad g(x) = \sin^{-1}(\sin x)$$

both simplify to just x for suitable values of x. But these functions are not the same for all x. Sketch graphs of both f and g to show how the functions differ. (Think carefully about the domain and range of $\sin^{-1}$.)

7.5 TRIGONOMETRIC EQUATIONS

A *trigonometric equation* is an equation that contains trigonometric functions. For example,

$$\sin^2 x + \cos^2 x = 1 \qquad \text{and} \qquad 2\sin x - 1 = 0$$

are both trigonometric equations. The first equation is an identity—that means it is true for every value of the variable x. The second equation is true only for certain values of x. In this section we are interested in solving trigonometric equations, that is, in finding all values of the variable that make the equation true.

EXAMPLE 1 ■ Solving a Trigonometric Equation

Solve the equation $2\sin x - 1 = 0$.

SOLUTION We first solve the equation for $\sin x$.

$$2\sin x - 1 = 0 \qquad \text{Given equation}$$

$$2\sin x = 1 \qquad \text{Add 1}$$

$$\sin x = \tfrac{1}{2} \qquad \text{Divide by 2}$$

In the interval $[0, 2\pi)$, the values of x for which this equation is true are $x = \pi/6$ and $x = 5\pi/6$. But since the sine function is periodic with period 2π, adding any integer multiple of 2π to these values gives another solution. Thus, all the solutions are of the form

$$x = \frac{\pi}{6} + 2k\pi \qquad \text{or} \qquad x = \frac{5\pi}{6} + 2k\pi$$

for any integer k. Figure 1 shows a graphical representation of the solutions.

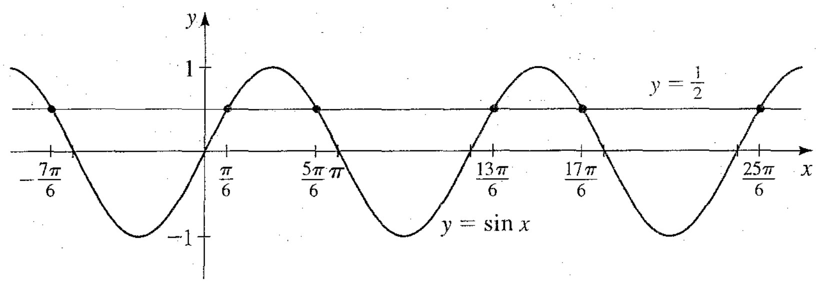


FIGURE 1 ■

In general, as in Example 1, if a trigonometric equation has one solution, then it has infinitely many solutions. To find all the solutions of such an equation, we need only find the solutions in an appropriate interval and then use the fact that the trigonometric functions are periodic.

EXAMPLE 2 ■ Solving by Factoring

Solve the equation $2\cos^2 x - 7\cos x + 3 = 0$.

SOLUTION We factor the left-hand side of the equation.

$$2\cos^2 x - 7\cos x + 3 = 0 \qquad \text{Given equation}$$

$$(2\cos x - 1)(\cos x - 3) = 0 \qquad \text{Factor}$$

$$2\cos x - 1 = 0 \quad \text{or} \quad \cos x - 3 = 0$$

$$\cos x = \tfrac{1}{2} \qquad\qquad \cos x = 3$$

$2C^2 - 7C + 3 = 0$

$(2C - 1)(C - 3) = 0$

Since $\cos x$ is never greater than 1, we see that $\cos x = 3$ has no solution. In the interval $[0, 2\pi)$, the equation $\cos x = \frac{1}{2}$ has solutions $x = \pi/3$ and $x = 5\pi/3$. Because the cosine function is periodic with period 2π, all the solutions are of the form

$$x = \frac{\pi}{3} + 2k\pi \quad \text{or} \quad x = \frac{5\pi}{3} + 2k\pi$$

for any integer k. ■

EXAMPLE 3 ■ An Equation Involving a Double Argument

Solve the trigonometric equation $\tan^4 2x - 9 = 0$.

SOLUTION

$$\tan^4 2x - 9 = 0 \qquad \text{Given equation}$$

$$\tan^4 2x = 9 \qquad \text{Add 9}$$

$$\tan 2x = \sqrt{3} \quad \text{or} \quad \tan 2x = -\sqrt{3} \qquad \text{Take fourth roots}$$

The interval $(-\pi/2, \pi/2)$ contains one complete period of the tangent function. In this interval, we have

$$2x = \frac{\pi}{3} \quad \text{or} \quad 2x = -\frac{\pi}{3}$$

Since the tangent function is periodic with period π, all the solutions are given by

$$2x = \frac{\pi}{3} + k\pi \quad \text{or} \quad 2x = -\frac{\pi}{3} + k\pi$$

$$x = \frac{\pi}{6} + \frac{k\pi}{2} \qquad x = -\frac{\pi}{6} + \frac{k\pi}{2} \qquad \text{Divide by 2}$$

for any integer k. ■

Trigonometric identities are useful tools for solving trigonometric equations. They can be used to transform an equation into an equivalent equation that's simpler to solve.

EXAMPLE 4 ■ Using a Trigonometric Identity

Solve the equation $3\sin x = 2\cos^2 x$ in the interval $[0, 2\pi)$.

SOLUTION Using the identity $\cos^2 x = 1 - \sin^2 x$, we get an equivalent equation that involves only the sine function.

$$3\sin x = 2\cos^2 x \qquad \text{Given equation}$$

$$3\sin x = 2(1 - \sin^2 x) \qquad \text{Pythagorean identity}$$

$$2\sin^2 x + 3\sin x - 2 = 0 \qquad \text{Simplify}$$

$$(2\sin x - 1)(\sin x + 2) = 0 \qquad \text{Factor}$$

$$2\sin x - 1 = 0 \quad \text{or} \quad \sin x + 2 = 0$$

$$\sin x = \tfrac{1}{2} \qquad \sin x = -2$$

Since $-1 \leqslant \sin x \leqslant 1$, the equation $\sin x = -2$ has no solution. The solutions of the given equation are thus the solutions of $\sin x = \frac{1}{2}$, that is, $x = \pi/6, 5\pi/6$. ■

EXAMPLE 5 ■ Finding Intersection Points

Find the points of intersection of the graphs of $f(x) = \sin x$ and $g(x) = \cos x$.

SOLUTION The graphs of f and g are shown in Figure 2.

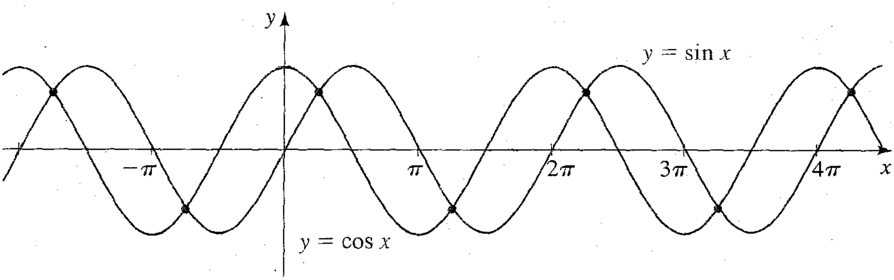


FIGURE 2

The graphs intersect at the points where $f(x) = g(x)$. So we need to find the solutions of the equation

$$\sin x = \cos x \qquad \text{Equate functions}$$

Notice that the numbers x for which $\cos x = 0$ are not solutions of this equation. For $\cos x \neq 0$, we can divide both sides of the equation by $\cos x$ to get

$$\frac{\sin x}{\cos x} = 1 \qquad \text{Divide by } \cos x$$

$$\tan x = 1 \qquad \text{Reciprocal identity}$$

The solution of this last equation in the interval $(-\pi/2, \pi/2)$ is $x = \pi/4$. Since the tangent function is periodic with period π, the solutions are

$$x = \frac{\pi}{4} + k\pi$$

for any integer k. ■

EXAMPLE 6 ■ Finding Solutions in an Interval

Find the solutions of $\cos 3x \sec x = 2\cos 3x$ in the interval $[0, 2\pi)$.

SOLUTION

$$\begin{aligned} \cos 3x \sec x &= 2\cos 3x && \text{Given equation} \\ \cos 3x \sec x - 2\cos 3x &= 0 && \text{Subtract } 2\cos 3x \\ \cos 3x(\sec x - 2) &= 0 && \text{Factor} \\ \cos 3x = 0 \quad &\text{or} \quad \sec x = 2 \end{aligned}$$

We begin by solving $\cos 3x = 0$. Since we are seeking solutions in the interval $0 \leqslant x \leqslant 2\pi$, we have $0 \leqslant 3x \leqslant 6\pi$. In this interval, $\cos 3x = 0$ has the solutions

$$3x = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2}, \frac{9\pi}{2}, \frac{11\pi}{2}$$

so

$$x = \frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{3\pi}{2}, \frac{11\pi}{6}$$

Now we solve $\sec x = 2$. In the interval $[0, 2\pi]$, the solutions of $\sec x = 2$ are $x = \pi/3$ and $x = 5\pi/3$. Thus, all the solutions of the given equation are

$$x = \frac{\pi}{6}, \frac{\pi}{3}, \frac{\pi}{2}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{3\pi}{2}, \frac{5\pi}{3}, \frac{11\pi}{6}$$ ■

⊘ Notice that in Example 6 we did not divide both sides by $\cos 3x$. It would have been wrong to do so, since we could be dividing by 0. Indeed, if we divide both sides by $\cos 3x$, then we lose all solutions of the given equation that are solutions of $\cos 3x = 0$.

EXAMPLE 7 ■ Taking Care of Extraneous Solutions

Solve the equation $\cos x + 1 = \sin x$ in the interval $[0, 2\pi)$.

SOLUTION To get an equation that involves either sine only or cosine only, we square both sides and use the Pythagorean identities.

$$\begin{aligned} \cos x + 1 &= \sin x && \text{Given equation} \\ (\cos x + 1)^2 &= \sin^2 x && \text{Square both sides} \\ \cos^2 x + 2\cos x + 1 &= \sin^2 x && \text{Expand} \\ \cos^2 x + 2\cos x + 1 &= 1 - \cos^2 x && \text{Pythagorean identity} \end{aligned}$$

$$2\cos^2 x + 2\cos x = 0 \qquad \text{Simplify}$$

$$2\cos x(\cos x + 1) = 0 \qquad \text{Factor}$$

$$\cos x = 0 \quad \text{or} \quad \cos x + 1 = 0$$

From these equations we get the possible solutions

$$x = \frac{\pi}{2}, \ \frac{3\pi}{2}, \ \pi \qquad \text{Potential solutions}$$

Since we may have introduced extraneous roots by squaring both sides of the equation, we must check to see whether each of these values for x satisfies the original equation. From *Check Your Answers,* we see that the solutions of the given equation in the interval $[0, 2\pi)$ are $\pi/2$ and π.

CHECK YOUR ANSWERS

$x = \frac{\pi}{2}$:	$x = \frac{3\pi}{2}$:	$x = \pi$:
$\cos\frac{\pi}{2} + 1 \stackrel{?}{=} \sin\frac{\pi}{2}$	$\cos\frac{3\pi}{2} + 1 \stackrel{?}{=} \sin\frac{3\pi}{2}$	$\cos\pi + 1 \stackrel{?}{=} \sin\pi$
$0 + 1 = 1$ ✓	$0 + 1 \stackrel{?}{=} -1$ ✗	$-1 + 1 = 0$ ✓

■

If we perform an operation on an equation that may introduce new roots, then we must check that the solutions obtained are not extraneous; that is, we must verify that they satisfy the original equation, as in Example 7.

EXAMPLE 8 ■ Using a Trigonometric Identity

Solve the equation: $\sin 2x - \cos x = 0$

SOLUTION Since the argument in the two terms is different, we first use the double-angle formula for sine.

$$\sin 2x - \cos x = 0 \qquad \text{Given equation}$$

$$2\sin x\cos x - \cos x = 0 \qquad \text{Double-angle formula}$$

$$\cos x(2\sin x - 1) = 0 \qquad \text{Factor}$$

$$\cos x = 0 \quad \text{or} \quad 2\sin x - 1 = 0$$

$$\sin x = \tfrac{1}{2}$$

For x in the interval $[0, 2\pi)$, the first of these equations has solutions $x = \pi/2, 3\pi/2$, and the second has solutions $x = \pi/6, 5\pi/6$. Thus, the solutions of the original equation are

$$x = \frac{\pi}{2} + 2k\pi, \quad \frac{3\pi}{2} + 2k\pi, \quad \frac{\pi}{6} + 2k\pi, \quad \frac{5\pi}{6} + 2k\pi$$

for any integer k. ■

USING A CALCULATOR TO SOLVE TRIGONOMETRIC EQUATIONS

So far all the equations we've solved have had solutions like $\pi/4$, $\pi/3$, $5\pi/6$, $3\pi/2$, and so on. We were able to find these solutions from the special values of the trigonometric functions that we've memorized. Now let's consider examples whose solutions require us to use the inverse trigonometric functions on a calculator.

EXAMPLE 9 ■ Using the Inverse Trigonometric Functions and a Calculator

Solve the equation $\sin x = \frac{2}{3}$ in the interval $[0, 2\pi]$.

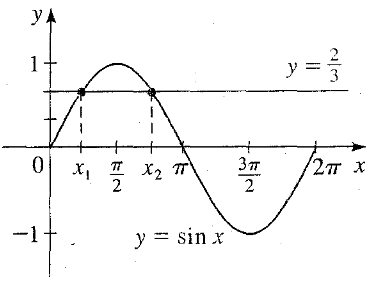


FIGURE 3

SOLUTION The solutions of this equation are the numbers x whose sine is $\frac{2}{3}$. From Figure 3 we see that there are two such numbers, x_1 and x_2, in the interval $[0, 2\pi]$. Using a calculator in radian mode, we can get x_1:

$\sin x = \frac{2}{3}$		Given equation
$x_1 = \sin^{-1}\frac{2}{3}$		Take $\sin^{-1}$ of each side
≈ 0.72973		Use a calculator

Note that since the range of $\sin^{-1}$ is the interval $[-\pi/2, \pi/2]$, the calculator gives us a number in that interval. From Figure 3 we see that the other solution lies between $\pi/2$ and π. (In fact, if we think of x as an angle in radian measure, then x_2 is the angle in the second quadrant whose reference angle is x_1.) Thus

$$x_2 = \pi - \sin^{-1}\tfrac{2}{3} \approx 2.41186$$

The solutions, correct to five decimal places are $x \approx 0.72973$ and $x \approx 2.41186$. ■

If we are using inverse trigonometric functions to solve an equation, we must keep in mind that $\sin^{-1}$ and $\tan^{-1}$ give values in quadrants I and IV, and $\cos^{-1}$ gives values in quadrants I and II. To find other solutions, we must look at the quadrant where the trigonometric function in the equation can take on the value we need.

EXAMPLE 10 ■ Finding All Solutions

Find all solutions of the equation $4\cos^2 x - 9\cos x + 2 = 0$.

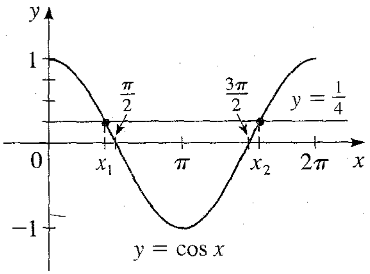


FIGURE 4

SOLUTION We first find the solutions in the interval $[0, 2\pi)$, which is one complete period of the cosine function.

$$\begin{aligned} 4\cos^2 x - 9\cos x + 2 &= 0 && \text{Given equation} \\ (\cos x - 2)(4\cos x - 1) &= 0 && \text{Factor} \end{aligned}$$

$$\cos x - 2 = 0 \quad \text{or} \quad 4\cos x - 1 = 0$$

$$\cos x = 2 \qquad\qquad \cos x = \tfrac{1}{4}$$

Since $\cos x$ cannot be larger than 1, the equation $\cos x = 2$ has no solution. From Figure 4 we see that the equation $\cos x = \frac{1}{4}$ has two solutions in the interval $[0, 2\pi)$. One of them is $x_1 = \cos^{-1}\frac{1}{4} \approx 1.31812$. Note that since the range of $\cos^{-1}$ is the interval $[0, \pi]$, the calculator gives us a number in that interval. The other solution is between $3\pi/2$ and 2π (the fourth quadrant), so its value is $x_2 = 2\pi - \cos^{-1}\frac{1}{4} \approx 4.96507$. Thus, all the solutions of the equation are of the form

$$x \approx 1.31812 + 2k\pi \quad \text{or} \quad x \approx 4.96507 + 2k\pi$$

for any integer k. ■

7.5 EXERCISES

1–30 ■ Find all solutions of the equation.

1. $2\cos x - 1 = 0$

2. $\sqrt{2}\sin x - 1 = 0$

3. $2\sin x - \sqrt{3} = 0$

4. $\tan x + 1 = 0$

5. $4\cos^2 x - 1 = 0$

6. $2\cos^2 x - 1 = 0$

7. $\sec^2 x - 2 = 0$

8. $\csc^2 x - 4 = 0$

9. $\cos x(2\sin x + 1) = 0$

10. $\sec x(2\cos x - \sqrt{2}) = 0$

11. $(\tan x + \sqrt{3})(\cos x + 2) = 0$

12. $(2\cos x + \sqrt{3})(2\sin x - 1) = 0$

13. $\cos x \sin x - 2\cos x = 0$

14. $\tan x \sin x + \sin x = 0$

15. $4\cos^2 x - 4\cos x + 1 = 0$

16. $2\sin^2 x - \sin x - 1 = 0$

17. $\sin^2 x = 2\sin x + 3$

18. $3\tan^3 x = \tan x$

19. $\sin^2 x = 4 - 2\cos^2 x$

20. $2\cos^2 x + \sin x = 1$

21. $2\sin 3x - 1 = 0$

22. $\sqrt{3}\sin 2x = \cos 2x$

23. $\cos\frac{x}{2} - 1 = 0$

24. $\csc 3x = \sin 3x$

25. $\tan^5 x - 9\tan x = 0$

26. $3\tan^3 x - 3\tan^2 x - \tan x + 1 = 0$

27. $4\sin x\cos x + 2\sin x - 2\cos x - 1 = 0$

28. $\sin 2x = 2\tan 2x$

29. $\cos^2 2x - \sin^2 2x = 0$

30. $\sec x - \tan x = \cos x$

31–38 ■ Find all solutions of the equation in the interval $[0, 2\pi)$.

31. $2\cos 3x = 1$

32. $3\csc^2 x = 4$

33. $2\sin x\tan x - \tan x = 1 - 2\sin x$

34. $\sec x\tan x - \cos x\cot x = \sin x$

35. $\tan x - 3\cot x = 0$

36. $2\sin^2 x - \cos x = 1$

37. $\tan 3x + 1 = \sec 3x$

38. $3\sec^2 x + 4\cos^2 x = 7$

39–46 ■ (a) Use a calculator to solve the equation in the interval $[0, 2\pi]$, correct to five decimal places.
(b) Find all solutions of the equation.

39. $\cos x = 0.4$

40. $2\tan x = 13$

41. $\sec x - 5 = 0$

42. $3 \sin x = 7 \cos x$

43. $5 \sin^2 x - 1 = 0$

44. $2 \sin 2x - \cos x = 0$

45. $3 \sin^2 x - 7 \sin x + 2 = 0$

46. $\tan^4 x - 13 \tan^2 x + 36 = 0$

47–50 ■ Sketch the graphs of f and g on the same axes, and find their points of intersection.

47. $f(x) = 3 \cos x + 1, \quad g(x) = \cos x - 1$

48. $f(x) = \sin 2x, \quad g(x) = 2 \sin 2x + 1$

49. $f(x) = \tan x, \quad g(x) = \sqrt{3}$

50. $f(x) = \sin x - 1, \quad g(x) = \cos x$

51–54 ■ Use an addition or subtraction formula to simplify the equation. Then find all solutions in the interval $[0, 2\pi)$.

51. $\cos x \cos 3x - \sin x \sin 3x = 0$

52. $\cos x \cos 2x + \sin x \sin 2x = \frac{1}{2}$

53. $\sin 2x \cos x + \cos 2x \sin x = \sqrt{3}/2$

54. $\sin 3x \cos x - \cos 3x \sin x = 0$

55–58 ■ Use a double- or half-angle formula to solve the equation in the interval $[0, 2\pi)$.

55. $\sin 2x + \cos x = 0$

56. $\tan \dfrac{x}{2} - \sin x = 0$

57. $\cos 2x + \cos x = 2$

58. $\tan x + \cot x = 4 \sin 2x$

59–62 ■ Solve the equation by first using a sum-to-product formula.

59. $\sin x + \sin 3x = 0$

60. $\cos 5x - \cos 7x = 0$

61. $\cos 4x + \cos 2x = \cos x$

62. $\sin 5x - \sin 3x = \cos 4x$

63–68 ■ Use a graphing device to find the solutions of the equation, correct to two decimal places.

63. $\sin 2x = x$

64. $\cos x = \dfrac{x}{3}$

65. $2^{\sin x} = x$

66. $\sin x = x^3$

67. $\dfrac{\cos x}{1 + x^2} = x^2$

68. $\cos x = \frac{1}{2}(e^x + e^{-x})$

DISCOVERY • DISCUSSION

69. Equations and Identities Which of the following statements is true?

A. Every identity is an equation.

B. Every equation is an identity.

Give examples to illustrate your answer. Write a short paragraph to explain the difference between an equation and an identity.

70. A Special Trigonometric Equation What makes the equation $\sin(\cos x) = 0$ different from all the other equations we've looked at in this section? Find all solutions of this equation.

7.6 TRIGONOMETRIC FORM OF COMPLEX NUMBERS; DeMOIVRE'S THEOREM

Complex numbers were introduced in Chapter 3 in order to solve certain algebraic equations. The applications of complex numbers go far beyond this initial use, however. Complex numbers are now used routinely in physics, electrical engineering, aerospace engineering, and many other fields. In this section we represent complex numbers using the functions sine and cosine. This will enable us to find the nth roots of complex numbers.