In the first section of Chapter 4, students consider different slope triangles for a given line or segment and notice that for each line, the slope remains constant no matter where they draw the slope triangle on that line or how large or small each slope triangle is. All the slope triangles on a given line are similar. These similar slope triangles allow students to write proportions to calculate lengths of sides and angle measures. This constant slope ratio is known as the "**tangent**" (trigonometric) relationship. Using the tangent button on their calculators, students are able to find measurements in application problems.

See the Math Notes boxes in Lessons 4.1.1, 4.1.2, and 4.1.4 for more information about slope angles and the tangent ratio.

Example 1

The line graphed at right passes through the origin. Draw in three different slope triangles for the line. For each triangle, what is the slope ratio, $\frac{\Delta y}{\Delta x}$? What is true about all three ratios?

Note: Δx (delta x) and Δy (delta y) are read "change in x" and "change in y."

A slope triangle is a right triangle that has its hypotenuse on the line that contains it. This means that the two legs of the right triangle are parallel to the axes: one leg runs vertically, the other horizontally. There are infinitely many slope triangles that we can draw, but it is always easiest if we draw triangles that have their vertices on lattice points (that is, their vertices have integer coordinates). The length of the horizontal leg is Δx and the length of the vertical leg is Δy. At right are three possible slope triangles. For the smallest triangle, $\Delta x = 3$ (the length of the horizontal leg), and $\Delta y = 2$ (the length of the vertical leg). For the smallest triangle we have $\frac{\Delta y}{\Delta x} = \frac{2}{3}$.

In the medium sized triangle, $\Delta x = 6$ and $\Delta y = 4$, which means $\frac{\Delta y}{\Delta x} = \frac{4}{6}$.

Lastly, the lengths on the largest triangle are $\Delta x = 15$ and $\Delta y = 10$, so $\frac{\Delta y}{\Delta x} = \frac{10}{15}$.

If we reduce the ratios to their lowest terms we find that the slope ratios, no matter where we draw the slope triangles for this line, are all equal.

$$\frac{\Delta y}{\Delta x} = \frac{2}{3} = \frac{4}{6} = \frac{10}{15}$$

Core Connections Geometry
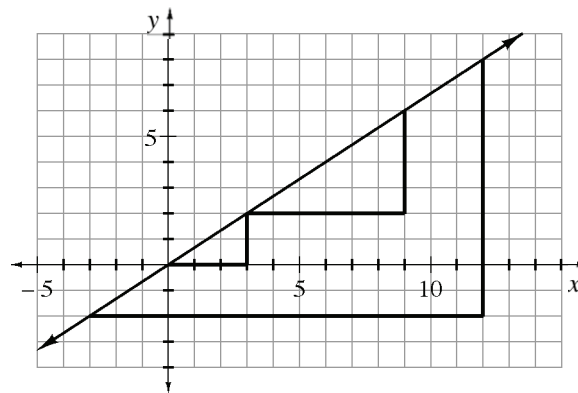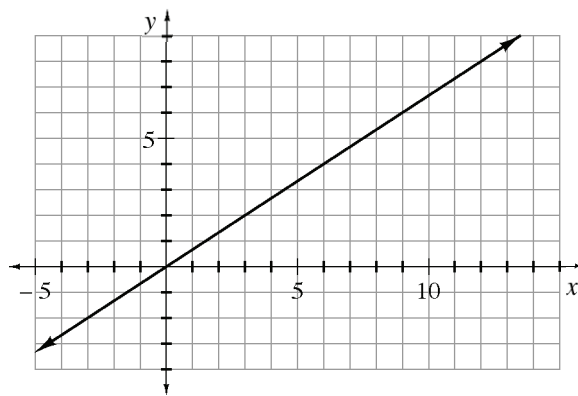
Students also discovered that different non-parallel lines do not have the same slope and slope ratio:the steeper the line, the larger the slope ratio, and the flatter the line, the smaller the slope ratio. In Lesson 4.1.2 students connect specific slope ratios to their related angles and record their findings in a Trig Table Toolkit (Lesson 4.1.2 Resource Page). They use this information to find missing side lengths and angle measures of right triangles. At the end of the section students use the tangent button on their calculators to find missing information in right triangles.
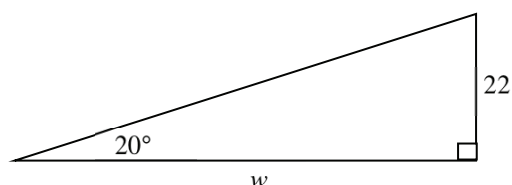
Example 2

Write an equation and use the tangent button on your calculator rather than your Trig Table Toolkit, to calculate the missing side length in each triangle.

a.
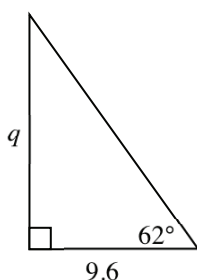
b.

When using the tangent button on a calculator with these problems, you must be sure that the calculator is in degree mode and not radian mode. Student should be able to check this and fix it, if necessary. Since we found that the slope ratio depends on the angle, we can use the angle measure and the tangent button on the calculator to find unknown lengths of the triangle.

In part (a), we know that the tangent of the angle is the ratio $\frac{\text{opposite leg}}{\text{adjacent leg}} = \frac{\Delta y}{\Delta x}$. This allows us to write the equation at right and solve it. Using a calculator, the value of "tan 62°" is ≈ 1.88.

$$\tan 62° = \frac{q}{9.6} \left(\frac{\text{opposite leg}}{\text{adjacent leg}} \right)$$

$$9.6(\tan 62°) = q$$

$$q \approx (9.6)(1.88) \approx 18.048$$

In part (b) we will set up another equation similar to the previous one. This equation is slightly different from the one in our first example in that the variable is in the denominator rather than the numerator. This adds another step to the solution process. Some students might realize that they can rotate the triangle and use the 70° angle (which they would have to determine using the sum of the measures of the angles of the triangle) so that the unknown side length is in the numerator.

$$\tan 20° = \frac{22}{w}$$

$$w \tan 20° = 22$$

$$w = \frac{22}{\tan 20°}$$

$$w \approx 60.44$$

Example 3

Talula is standing 117 feet from the base of the Washington Monument in Washington, D.C. She uses her clinometer to measure the angle of elevation to the top of the monument to be 78°. If Talula's eye height is 5 feet, 3 inches, what is the height of the Washington Monument?
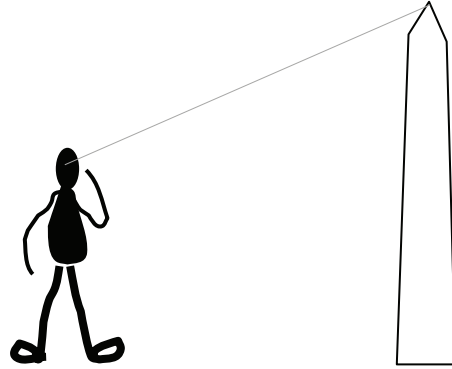
With all problems representing an everyday situation, the first step is the same: draw a picture of what the problem is describing. Here, we have Talula looking up at the top of a monument. We know how far away Talula is standing from the monument, we know her eye height, and we know the angle of elevation of her line of sight.

We translate this information from the picture to a diagram, as shown at right. On this diagram we include all the measurements we know. Then we write an equation using the tangent function and solve for x:
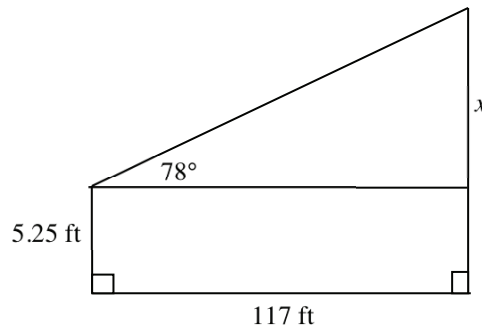
$$\tan 78° = \tfrac{x}{117}$$
$$117(\tan 78°) = x$$
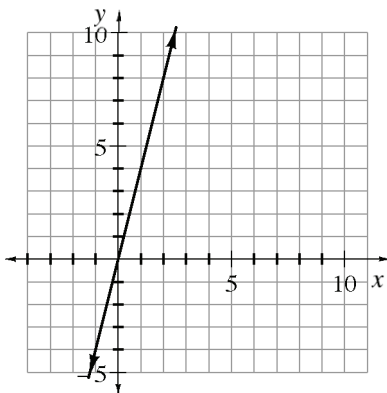$$x \approx 549.9 \text{ feet}$$

We add the "eye height" to the value of x to find the height of the Washington Monument:

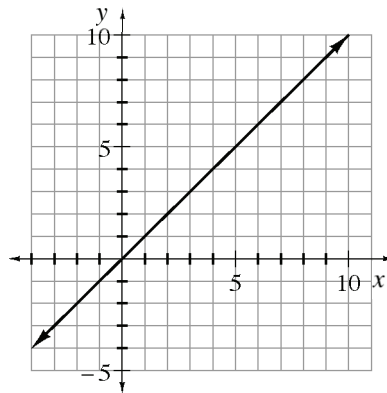$$549.9 + 5.25 \approx 555.15 \text{ feet}$$

Problems

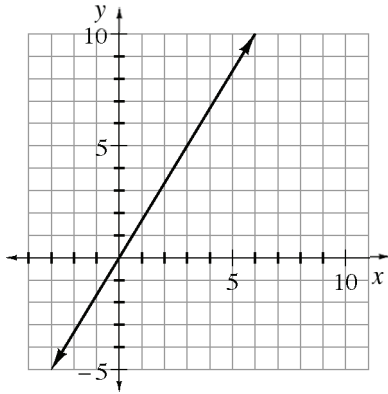For each line, draw in several slope triangles. Then calculate the slope ratios.
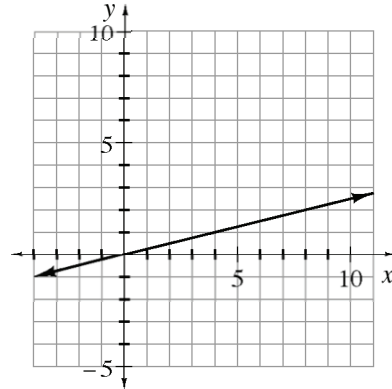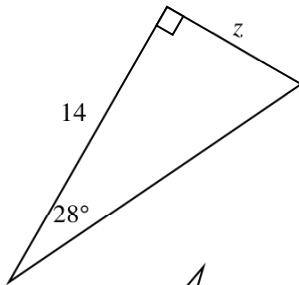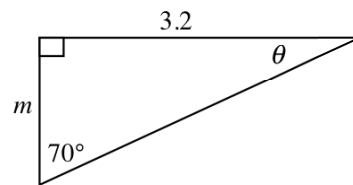
1.

2.

3.

4.

Calculate the measures of the variables. It may be helpful to rotate the triangle so that it resembles a slope triangle. If you write a tangent equation, use the tangent button on your calculator not your Trig Toolkit to solve. Note: Some calculations require the Pythagorean Theorem.
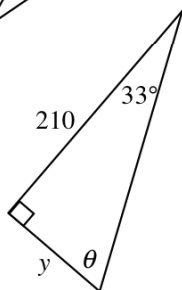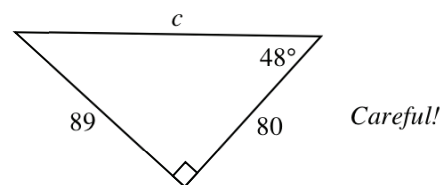
5.

z

14

28°

6.

3.2

θ

m

70°

7.

33°

210

y θ

8.

c

48°

89

80

Careful!

9.

θ

x

45°

12.25

10.

w

15°

47
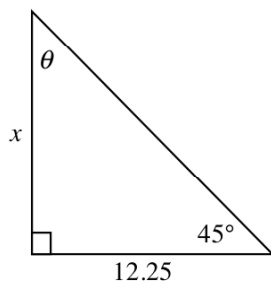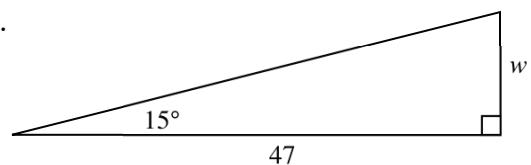
11. A ladder makes a 75° angle with the wall it is leaning against. The base of the ladder is 5 feet from the wall. How high up the wall does the ladder reach?

12. Davis and Tess are 30 feet apart when Tess lets go of her helium-filled balloon, which rises straight up into the air. (It is a windless day.) After 4 seconds, Davis uses his clinometer to site the angle of elevation to the balloon at 35°. If Davis' eye height is 4 feet, 6 inches, what is the height of the balloon after 4 seconds?

Answers

1. In each case the slope ratio is $\frac{4}{1} = 4$.
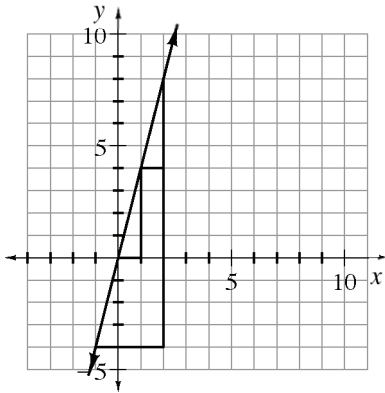
2. The slope ratio is $\frac{5}{5} = \frac{4}{4} = \frac{3}{3} = \frac{1}{1}$.
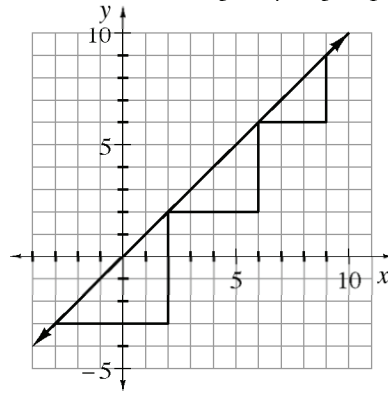
3. The slope ratio is $\frac{5}{3}$.
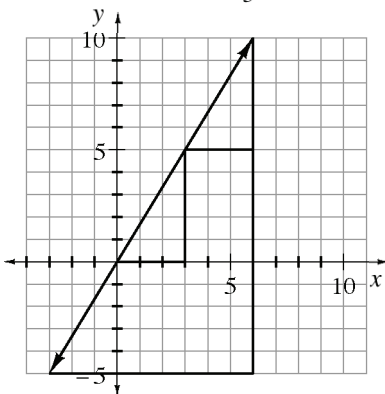
4. The slope ratio is $\frac{1}{4}$.
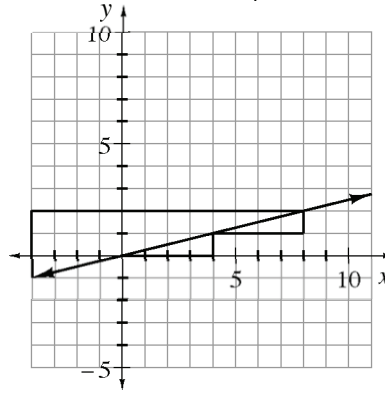
5. $\tan 28° = \frac{z}{14}$, $z \approx 7.44$

6. $\tan 70° = \frac{3.2}{m}$, $m \approx 2.75$, $\theta = 20°$

7. $\tan 33° = \frac{y}{210}$, $y \approx 136.38$, $\theta = 57°$

8. $c \approx 119.67$ (Pythagorean Theorem)

9. $\theta = 45°$, $x = 12.25$

10. $\tan 15° = \frac{w}{47}$, $w \approx 12.59$

11. $\tan 75° = \frac{h}{5}$; the ladder reaches about 18.66 feet up the wall.

12. $\tan 35° = \frac{h}{30}$, $h \approx 21 + 4.5 \approx 25.5$; After 4 seconds the balloon is about 25.5 feet above the ground.

Core Connections Geometry
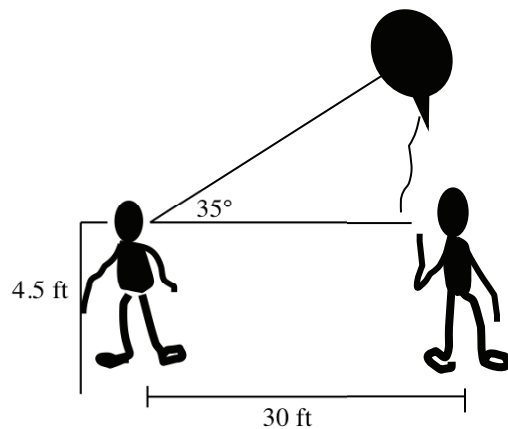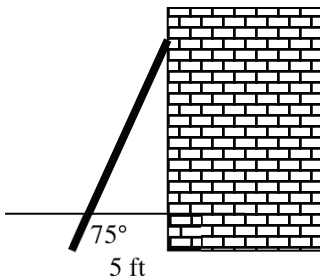
Although the definition of probability is simple, calculating a particular probability can sometimes be tricky. When calculating the probability of flipping a coin and having it come up tails, we can easily see that there are only two possibilities and one successful outcome. But what if neither the total number of outcomes nor the total number of successes is obvious? In this case, we need to have an accurate way to count the number of these events. In these lessons, we look at three models to do this: making a systematic list, making a tree diagram, and making an area model. Each different model has its strengths and weaknesses, and is more efficient in different situations.

See the Math Notes boxes in Lessons 4.2.3 and 4.2.4 for more information about calculating probabilities.

Example 1

As Ms. Dobby prepares the week's lunch menu for the students, she has certain rules that she must follow. She must have a meat dish and a vegetable at each lunch. She has four choices for meat: chicken, fish, beef, and pork. Her list of choices for vegetables is a bit larger: peas, carrots, broccoli, corn, potatoes, and beets. Considering just the meat and the vegetable, what is the probability that the first lunch she makes will have meat and a green vegetable?

To determine the probability of a lunch with meat and a green vegetable, we need to know how many different lunch menus are possible. Then we need to count how many of the lunch menus have meat and a green vegetable. To count all of the possible lunch menus, we will make a systematic list, pairing each meat with a vegetable in an organized way.

Chicken	Fish	Beef	Pork
Chicken and peas	Fish and peas	Beef and peas	Pork and peas
Chicken and carrots	Fish and carrots	Beef and carrots	Pork and carrots
Chicken and broccoli	Fish and broccoli	Beef and broccoli	Pork and broccoli
Chicken and corn	Fish and corn	Beef and corn	Pork and corn
Chicken and potatoes	Fish and potatoes	Beef and potatoes	Pork and potatoes
Chicken and beets	Fish and beets	Beef and beets	Pork and beets

From this list we can count the total number of lunch menus: 24. Then we count the number of lunch menus with meat and a green vegetable (peas or broccoli). There are 6 such menus. Therefore the probability of the first lunch menu having meat and a green vegetable is $\frac{6}{24} = \frac{1}{4}$.

Example 2

What is the probability of flipping a fair coin 4 times and have tails come up exactly two of those times?

To solve this problem, we could make a systematic list as we did in the previous example, but there is another technique that works well for this type of problem. Since each flip gives us only two outcomes, we can organize this information in a tree diagram. The first flip has only two possibilities: heads (H) or tails (T). From each branch, we split again into H or T. We do this for each flip of the coin. The final number of branches at the end tells us the total number of outcomes. In this problem, there are 16 outcomes. We now count the number of "paths" along the branches that have exactly two Ts. One path consisting of HTHT is highlighted. The others are HHTT, HTTH, THHT, THTH, and TTHH, for a total of six paths. Thus the probability of flipping a coin four times and having T come up exactly two times is $\frac{6}{16} = \frac{3}{8}$.
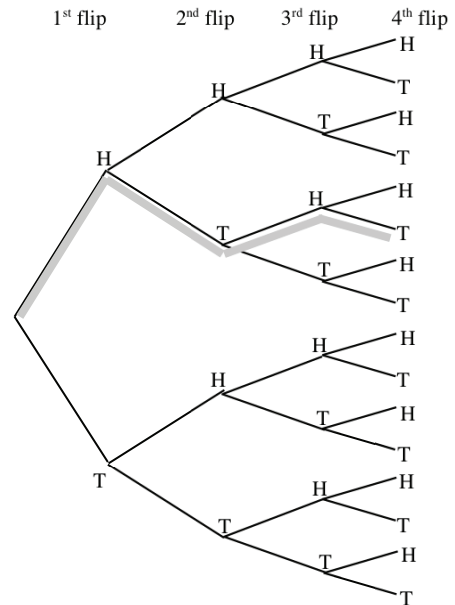
Example 3

Romeo the rat is going to run through a maze to find a block of cheese. The floor plan of the maze is shown at right, with the cheese to be placed in either section A or section B. If every time Romeo comes to a split in the maze he is equally like to choose any path in front of him, what is the probability he ends up in section A?
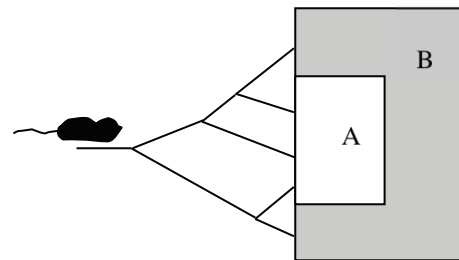
To answer this question we will construct an area model to represent this situation. Using an area model is like turning the problem into a dartboard problem. It is easy to see what the probability of hitting the shaded portion on the dartboard at right is because the shaded portion makes up one-fourth of the board. Therefore the probability of hitting the shaded portion is $\frac{1}{4}$. What we want to do is turn the maze problem into a dartboard with the outcome we want (our success) represented by the shaded part.
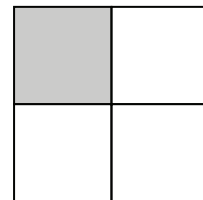
To begin, we start with a square dartboard. You can think of this as being a 1×1 square. When Romeo comes to the first branch in the maze, he has two choices: a top path and a bottom path. We represent this on the dartboard by splitting the board into two same sized (equally likely) pieces. Then consider what happens if Romeo chooses the bottom path first. If he chooses the bottom path, he comes to another split with two choices, each equally likely. On the area model (dartboard) we show this by splitting the bottom rectangle into two equally like sections, shown at right.
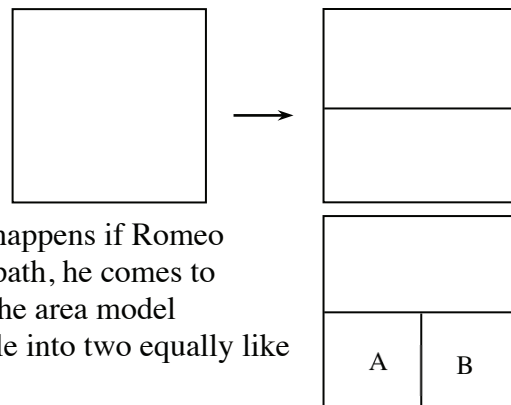
With one branch, Romeo will end up in section A; with the other branch he will end up in B. We indicate this by putting the letters in the regions representing these outcomes. Note: you can split the bottom rectangle in half with a "top" rectangle and a "bottom" rectangle as well. Since we are ultimately going to consider the area covered with an "A," it can be split in any way as long as the pieces are equal in size.

Now consider the top path. If Romeo takes the top path at the first split, he quickly comes to another split where again he has a choice of a top path or a bottom path. Once again we split the top rectangle into two same-sized rectangles since each path is equally likely. One box will represent the top path and one will represent the bottom. If Romeo takes the lower path, he will end up in section A. We indicate this by choosing one of the new regions as representing the lower path, and writing an A in that portion. If Romeo takes the upper path, he comes to another split, each equally likely. This means the last section of the dartboard that is not filled in needs to be cut into two equal parts, since each path is equally likely. One of the paths will lead directly to section A, the other to section B. Now we can fill in those letters as well.

By looking at the dart board now, we can see that since A takes up more of the board, we would be more likely to hit section A. But to find the actual probability, we must determine how much area the sections marked with A take up. Recall that this is a 1×1 square. We can find the fraction of the area of each part. Remember: the key is that we divided regions up into equal parts. The length of each side of each rectangle is shown on the exterior of the square, while the area is written within the region. We want to know the probability of getting into section A, which is represented by the shaded portion of the dartboard. The area of the shaded region is:
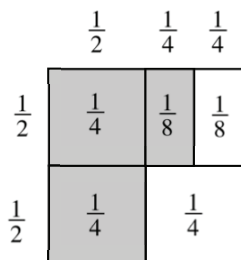
$$A = \tfrac{1}{4} + \tfrac{1}{4} + \tfrac{1}{8}$$
$$= \tfrac{2}{8} + \tfrac{2}{8} + \tfrac{1}{8}$$
$$= \tfrac{5}{8}$$

Therefore the probability of Romeo wandering into section A is $\tfrac{5}{8}$. This means the probability that he wanders into section B is $\tfrac{3}{8}$ since the sum of both probabilities must be 1.
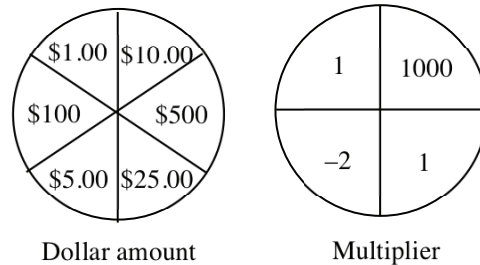
Problems

1. If Keisha has four favorite shirts (one blue, one green, one red, and one yellow) and two favorite pairs of pants (one black and one brown), how many different favorite outfits does she have? What is the best way to count this?

2. Each morning Aaron starts his day with either orange juice or apple juice followed by cereal, toast, or scrambled eggs. How many different morning meals are possible for Aaron?

3. Eliza likes to make daily events into games of chance. For instance, before she went to buy ice cream at the local ice cream parlor, she created two spinners. The first has her three favorite flavors while the second has "cone" and "dish." Eliza will order whatever comes up on the spinners. What is the probability that she will be eating tutti fruitti ice cream from a dish?
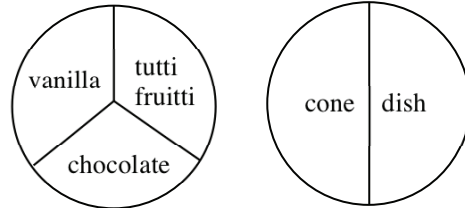

4. Barty is going to flip a coin three times. What is the probability that he will see *at least* two tails?

5. Mr. Fudge is going to roll two fair dice. What is the probability that the sum will be 4 or less?

6. Welcome to another new game show, "Spinning for Luck!" As a contestant, you will be spinning two wheels. The first wheel determines a possible dollar amount that you could win. The second wheel is the "multiplier." You will multiply the two results of your spin to determine the amount you will win. Unfortunately, you could *owe* money if your multiplier lands on –2! What is the probability that you could win $100 or more? What is the probability that you could owe $100 or more?


Dollar amount   Multiplier

For problems 7 through 10, a bag contains the figures shown below right. If you reach in and pull out a shape at random, what is the probability that you pull out:

7. A figure with at least one right angle?

8. A figure with an acute angle?

9. A shape with at least one pair of parallel sides?

10. A triangle?


Core Connections Geometry
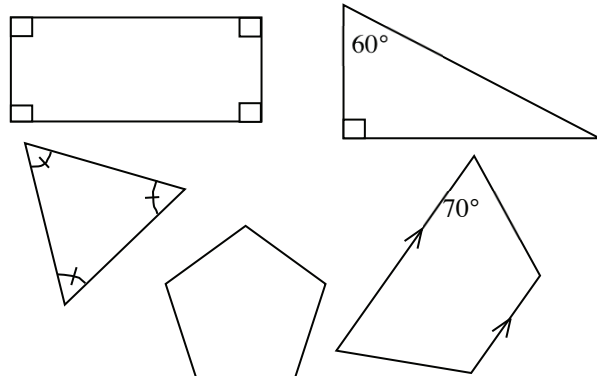
For each question that follows, use an area model or a tree diagram to compute the desired probability.

For problems 11-13 use the spinners at right.
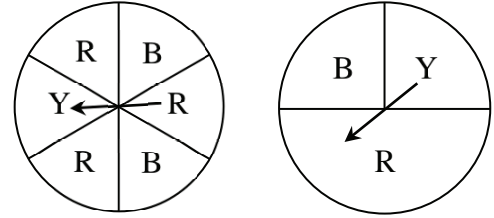
11. If each spinner is spun once, what is the probability that both spinners show blue?

12. If each spinner is spun once, what is the probability that both spinners show the same color?

13. If each spinner is spun once, what is the probability of getting a red-blue combination?

14. A pencil box has three yellow pencils, one blue pencil, and two red pencils. There are also two red erasers and one blue. If you randomly choose one pencil and one eraser, what is the probability of getting the red-red combination?

15. Sally's mother has two bags of candy but she says that Sally can only have one piece. Bag #1 has 70% orange candies and 30% red candies. Bag #2 has 10% orange candies, 50% white candies, and 40% green candies. Sally's eyes are covered and she chooses one bag and pulls out one candy. What is the probability that she chooses an orange candy?

16. You roll a die and flip a coin. What is the probability of rolling a number less than 5 on the die and flipping tails on the coin?

17. A spinner is evenly divided into eight sections—three are red, three are white, and two are blue. If the spinner is spun twice, what is the probability of getting the same color twice?

18. You and your friend have just won a chance to collect a million dollars. You place the money in one room at right and then your friend has to randomly walk through the maze. In which room should you place the money so that your friend will have the best chance of finding the million dollars?
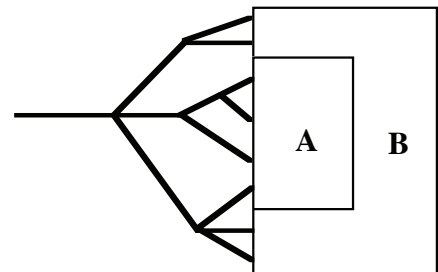
19. Find the probability of randomly entering each room in the maze shown at right.

   a.   P(A)   b.   P(B)   c.   P(C)
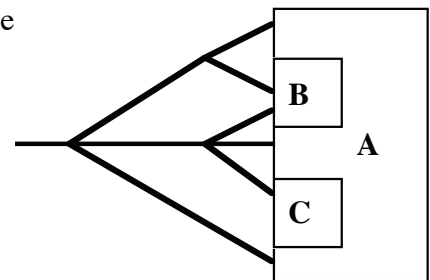
20. The weather forecast shows a 60% chance of rain. If it does not rain then there is an 80% chance of going to the beach. What is the probability of going to the beach

21. A baseball player gets a hit 40% of the time if the weather is nice but only 20% of the time if it is cold or windy. The weather forecast shows a 70% chance of being nice, 20% chance of being cold, and 10% chance of being windy. What is the probability of that the baseball player will get a hit?

22.   If students have their assignments done on time there is an 80% chance of earning a good grade in the class. If the assignments are finished during class or late then there is only a 30% chance of earning a good grade in the class. If the assignments are not done at all then there is only a 5% chance of earning a good grade in the class. In a certain class, 50% of the students have the assignments completed on time, 40% finish during class, and 10% do not do their assignments. If a student is selected at random, what is the probability that student has a good grade?

Answers

1.   Eight different outfits. A systematic list works best.

2.   Six meals. A systematic list or tree diagram works.

3. $\frac{1}{3} \cdot \frac{1}{2} = \frac{1}{6}$

4. $\frac{1}{2}$ (See the tree diagram in Example 2.)

5. $\frac{10}{36} = \frac{5}{18}$

6.   Winning $100 or more: $\frac{5}{12}$, owing $100 or more: $\frac{1}{12}$

7. $\frac{2}{5}$ 8. $\frac{3}{5}$ 9. $\frac{2}{5}$ 10. $\frac{2}{5}$

11. $\frac{1}{12}$ 12. $\frac{9}{24} = \frac{3}{8}$ 13. $\frac{7}{24}$ 14. $\frac{2}{9}$

15. $\frac{2}{5}$ 16. $\frac{1}{3}$ 17. $\frac{11}{32}$ 18. P(B) = $\frac{5}{9}$

19. $\frac{11}{18}$, $\frac{5}{18}$, $\frac{2}{18}$ 20. 0.32   21. 0.34   22. 0.525

Lesson 4.2.5 investigates expected value. Utilizing what students already know about probability, their study is extended, first through collecting data, and then by developing the mathematics behind the problem. Ultimately, students come up with a formula for calculating the expected value for each play of a game. Something to be careful of in this section: students sometimes think that the expected value must actually be one of the possible outcomes. It does not have to be. The expected value tells what is the average expected result for *one* play.

See the Math Notes box in Lesson 5.2.2 for more information about expected value.

Example 1

The spinner at right is divided into different sections, each assigned a different point value. The three smaller sections are congruent. If you were to spin the spinner 100 times, how many times would you expect to get each of the different point values? What is the expected value of this spinner?
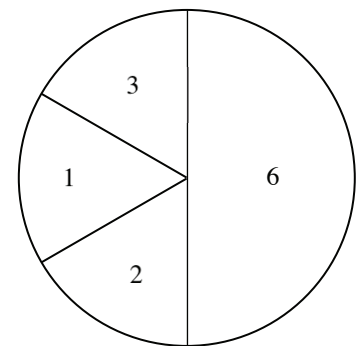
The size of each region is what determines the probability of the spinner landing in that region. Therefore the probability of landing on 6 points is $\frac{1}{2}$ because that region takes up half of the spinner. The other half of the circle is divided into three equal parts, each taking up $\frac{1}{6}$ of the whole spinner ($\frac{1}{3}$ of $\frac{1}{2}$). Now that we know the probabilities, we can determine how many times we would expect the values to come up. Since the probability of getting 6 points is $\frac{1}{2}$, we would expect that about half of the 100 spins would land on the 6, so 50 times. Similarly, since the probability of landing on 1 point (or 2 or 3) is $\frac{1}{6}$, would we expect about $\frac{1}{6}$ of the 100 spins to land on each of those, or about 16 or 17 times. If the total number of spins is 100, we can expect on average about 50 of them to be 6 points, $16\frac{2}{3}$ to be 1 point, $16\frac{2}{3}$ to be 2 points, and $16\frac{2}{3}$ to be 3 points. (Note: These are estimates, not exact or guaranteed.) Using these values, after 100 spins, the player would have about $50(6)+16\frac{2}{3}(1)+16\frac{2}{3}(2)+16\frac{2}{3}(3)=400$ points. If the player earns 400 points in 100 spins, then on average the player received 4 points per spin. So for any single spin, the expected value is 4 points. Note: 4 points is the expected value for this spinner, but it is NOT one of the possible outcomes.

Example 2

A 3×3 grid of nine congruent squares, each with a side length of 2 inches, is painted various colors. Six of the small squares are painted red while three are painted blue. For $1.00 a player can throw a dart at the grid. If the player hits a blue square, he is handed $2.00. Is this a fair game? Justify your answer.

The definition of a "fair" game is one in which the expected value is 0 because this means that, on average, the player is not guaranteed to win, and neither is the person running the game. If the expected value is 0, then winning or losing is just a matter of luck, and the game does not favor one side over the other. To determine if this game is fair we need to calculate its expected value.

Although we could go through a procedure similar to what we did in the last problem, there is a formula that is derived from that procedure that we can use. The expected value is found by summing the products of the amounts won and their probabilities. In this problem, each game costs $1.00 to play. If the dart lands on a red square, the player loses $1.00 (the value is -1). The probability of hitting a red square is $\frac{6}{9} = \frac{2}{3}$. However, if the player hits a blue square, the player receives $2.00, which wins only $1.00 (because he paid $1.00 for the dart). Based on the calculations at right, the expected value is $-\frac{1}{3}$. This tells us that on average the player can expect to lose $\frac{1}{3}$ of a dollar, or about $0.33, each turn. Therefore, this is not a fair game; it favors the person running the game.

$$E = \tfrac{2}{3}(-1.00) + \tfrac{1}{3}(1.00)$$
$$= -\tfrac{2}{3} + \tfrac{1}{3}$$
$$= -\tfrac{1}{3}$$

Example 3

In the previous problem, if we let Romeo run through the maze randomly 80 times, how many times would you expect him to end up in section A? In section B?

Now that we know the probability of Romeo wandering into each of the sections, we can figure out how many times we would *expect* him to reach each section. Since the probability of Romeo wandering into section A is $\frac{5}{8}$, we would expect Romeo to end up in section A 50 out of 80 times. Similarly, we would expect Romeo to wander into section B 30 times out of 80. This does not mean that Romeo will definitely wander into A 50 times out of 80. We are dealing with probabilities, not certainties, and this just gives us an idea of what to expect.
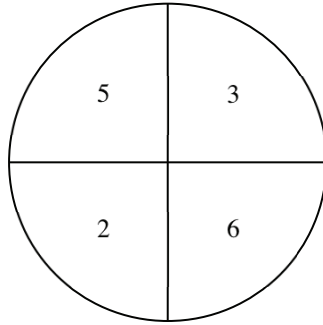
Problems

The spinners below have different point values assigned to the different regions. What is the expected value for each spinner? (Assume that regions that appear to be congruent, are congruent.)
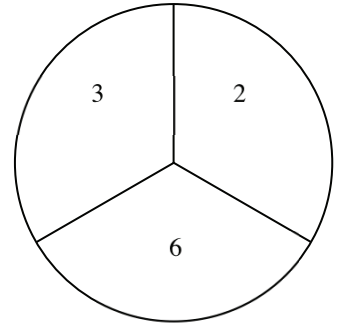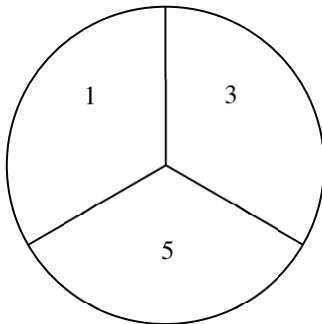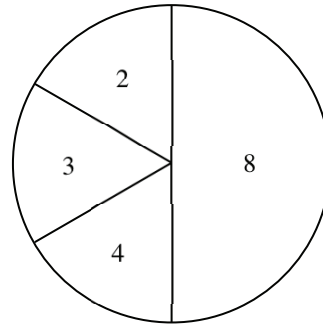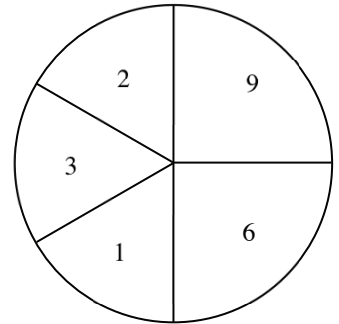
1.
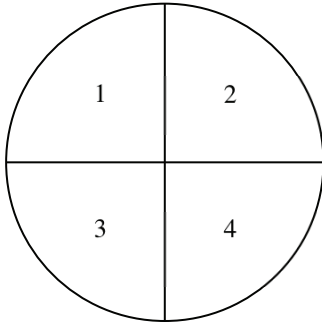
2.

3.

4.

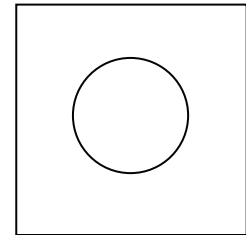5.

6.

7.   For $0.40 a player gets one dart to throw at a board that looks like the figure at right. The board is a square, measuring one foot along each side. The circle is centered and has a diameter of six inches. For each dart that lands in the interior of the circle, the players gets $0.75.
Is this game fair? Justify your answer.

Answers

1.	2.5	2.	4	3.	≈ 3.7
4.	3	5.	5.5	6.	4.75

7.   Not fair because the expected value is about $-\$1.25$.