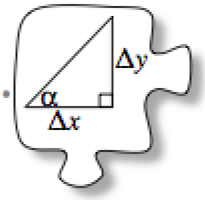
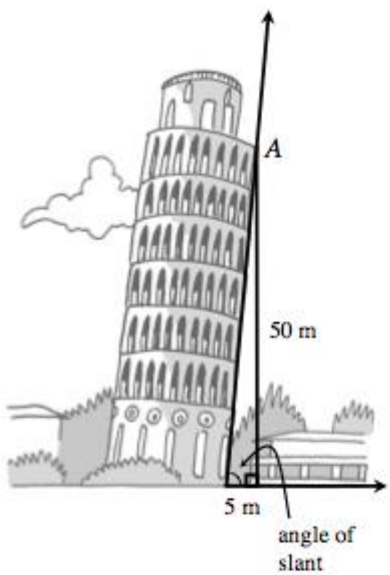
4.1.1 What patterns can I use?

Constant Ratios in Right Triangles



In Chapter 3, you looked for relationships between triangles and ways to determine if they are similar or congruent. Now you are going to focus your attention on slope triangles, which were used in algebra to describe linear change. Are there geometric patterns within slope triangles themselves that you can use to answer other questions? In this lesson, you will look closely at slope triangles on different lines to explore their patterns.



4-1. LEANING TOWER OF PISA

For centuries, people have marveled at the Leaning Tower of Pisa due to its slant and beauty. Ever since construction of the tower started in the 1100s, the tower has slowly tilted south and has increasingly been at risk of falling over. It is feared that if the angle of slant ever falls below 83°, the tower will collapse.

Engineers closely monitor the angle at which the tower leans. With careful measuring, they know that the point labeled *A* in the diagram at right is now 50 meters off the ground. Also, they determined that when a weight is dropped from point *A*, it lands 5 meters from the base of the tower, as shown in the diagram. Explore estimating the angle the tower is leaning using 4-1 Student eTool (Desmos).

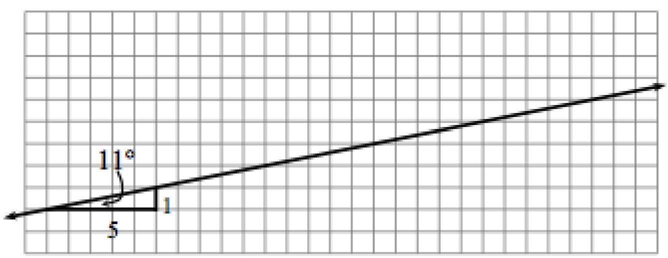
a. With the measurements provided above, what can you determine?

b. Can you determine the angle at which the tower leans? Why or why not?

c. At the end of Section 4.1, you will know how to find the angle for this situation and many others. However, at this point, how else can you describe the "lean" of the leaning tower?

4-2. PATTERNS IN SLOPE TRIANGLES

In order to find an angle (such as the angle at which the Leaning Tower of Pisa leans), you need to investigate the relationship between the angles and the sides of a right triangle. You will start by studying slope triangles. Obtain the Lesson 4.1.1 Resource Pages from your teacher and find the graph shown below. Notice that one slope triangle has been drawn for you. Note: For the next several lessons angle measures will be rounded to the nearest degree.



a. Draw three new slope triangles on the line. Each should be a different size. Label each triangle with as much information as you can, such as its horizontal and vertical lengths and its angle measures

b. Explain why all of the slope triangles on this line must be similar.

c. Since the triangles are similar, what does that tell you about the slope ratios?

d. Confirm your conclusion by writing the slope ratio for each triangle as a fraction, such as $\frac{\Delta y}{\Delta x}$. (Note: Δy represents the vertical change or "rise," while Δx represents the horizontal change or "run.") Then change the slope ratio into decimal form and compare.

4-3. Tara thinks she sees a pattern in these slope triangles, so she decides to make some changes in order to investigate whether or not the patterns remain true.

a. She asks, "*What if I draw a slope triangle on this line with* $\Delta y = 6$*? What would be the* Δx *of my triangle?*" Answer her question and explain how you figured it out.

b. "What if Δx is 40?" she wonders. "Then what is Δy?" Find Δy , and explain your reasoning.

c. Tara wonders, "What if I draw a slope triangle on a different line? Can I still use the same ratio to find a missing Δx or Δyvalue?" Discuss this question with your team and explain to Tara what she could expect.

4-4. CHANGING LINES

In part (c) of problem 4-3, Tara asked, "*What if I draw my triangle on a different line?*" With your team, investigate what happens to the slope ratio and slope angle when the line is different. Use the grids provided on your Lesson 4.1.1 Resource Pagesto graph the lines described below. Use the graphs and your answers to the questions below to respond to Tara's question.

a. On graph A, graph the line $y = \frac{2}{5}x$. What is the slope ratio for this line? What does the slope angle appear to be? Does the information about this line support or change your conclusion from part (c) of problem 4-3? Explain.

b. On graph B, you are going to create $\angle QPR$ so that it measures 18°. First, place your protractor so that point P is the vertex. Then find 18° and mark and label a new point, *R*. Draw ray $\overrightarrow{PR}$ to form $\angle QPR$. Find an approximate slope ratio for this line.

c. Graph the line $y = x + 4$ on graph C. Draw a slope triangle and label its horizontal and vertical lengths. What is $\frac{\Delta y}{\Delta x}$ (the slope ratio)? What is the slope angle?

4-5. TESTING CONJECTURES

The students in Ms. Coyner's class are writing conjectures based on their work today. As a team, decide if you agree or disagree with each of the conjectures below. Explain your reasoning.

- All slope triangles have a ratio $\frac{1}{5}$.

- If the slope ratio is $\frac{1}{5}$ then the slope angle is approximately 11°.

- If the line has an 11° slope angle, then the slope ratio is approximately $\frac{1}{5}$.

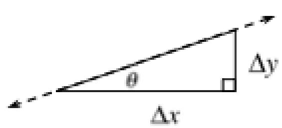
- Different lines will have different slope angles and different slope ratios.

METHODS AND MEANINGS

MATH NOTES

Slope and Angle Notation

The slope of a line is the ratio of the vertical distance to the horizontal distance in a slope triangle formed by two points on a line. The vertical part of the triangle is called Δy, (read "change in *y*"), while the horizontal part of the triangle is called Δx (read "change in *x*"). Slope can then be written as $\frac{\Delta y}{\Delta x}$. Slope indicates both how steep the line is and its direction, upward or downward, left to right.

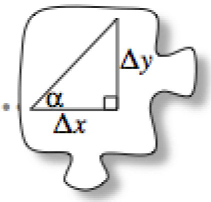


When a side length in a triangle is missing, that length is often assigned a variable from the English alphabet such as *x*, *y*, or *z*. However, sometimes you need to distinguish between an unknown side length and an unknown angle measure. With that in mind, mathematicians sometimes use Greek letters as variables for angle measurement. The most common variable for an angle is the Greek letter θ(*theta*), pronounced "THAY-tah." Two other Greek letters commonly used include α(*alpha*), and β(*beta*), pronounced "BAY-tah."

When a right triangle is oriented like a slope triangle, such as the one in the diagram above, the angle the line makes with the horizontal side of the triangle is called a **slope angle**.

4.1.2 How important is the angle?

Connecting Slope Ratios to Specific Angles



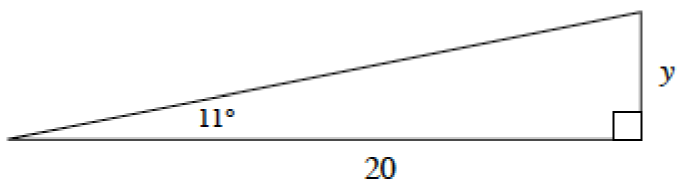
In Lesson 4.1.1, you started **trigonometry**, the study of the measures of triangles. As you continue to investigate right triangles with your team today, use the following questions to guide your discussion:

What do I know about this triangle?

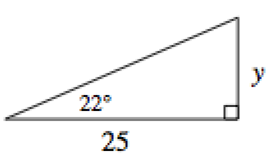
How does this triangle relate to other triangles?

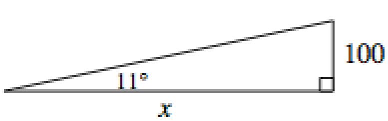
Which part is Δx? Which part is Δy?

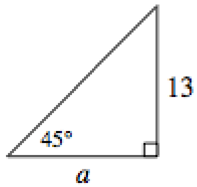
4-12. What do you know about this triangle? To what other triangles does it relate? Use any information you have to solve for y.

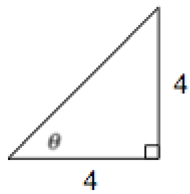


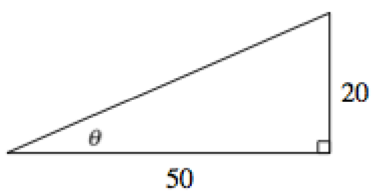
4-13. For each triangle below, find the missing angle or side length. Use your work from Lesson 4.1.1 to help you.

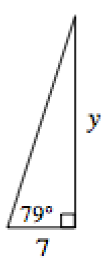
a. 

b. 

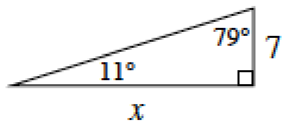
c. 

d. 

e. 

f. 

4-14. Sheila says the triangle in part (f) of problem 4-13 is the same as her drawing below.



a. Do you agree? Use tracing paper to convince you of your conclusion.

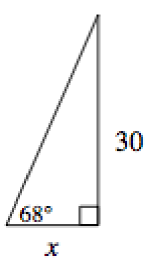
b. Use what you know about the slope ratio of 11° to find the slope ratio for 79°.

c. What is the relationship of 11° and 79°? Of their slope ratios?

4-15. For what other angles can you find the slope ratios based on the work you did in Lesson 4.1.1?

a. For example, since you know the slope ratio for 22°, what other angle do you know the slope ratio for? Use tracing paper to find a slope ratio for the complement of each slope angle you know. Use tracing paper to help re-orient the triangle if necessary.

b. Use this information to find *x* in the diagram below.



c. Write a conjecture about the relationship of the slope ratios for complementary angles. You may want to start with, "*If one angle of a right triangle has the slope ratio* $\frac{a}{b}$ *, then …*"

4-16. BUILDING A TRIGONOMETRY TABLE

So far you have looked at several similar slope triangles and their corresponding slope ratios. These relationships will be very useful for finding missing side lengths or angle measures of right triangles for the rest of this chapter.

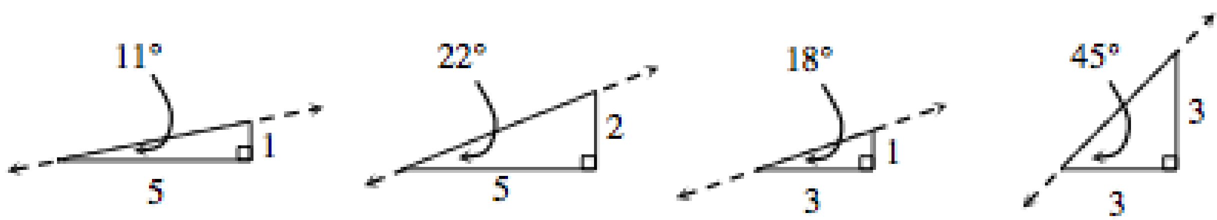
Before you forget this valuable information, organize information about the triangles and ratios you have discovered so far in the table on the Lesson 4.1.2 (“Trig Table Toolkit”) Resource Page provided by your teacher. Keep it in a safe place for future reference. Include all of the angles you have studied up to this point. An example for 11° is filled in on the table to get you started.

METHODS AND MEANINGS

MATH NOTES

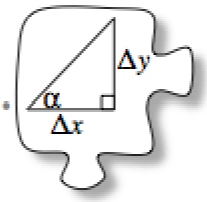
Slope Ratios and Angles

In Lesson 4.1.1, you discovered that certain slope angles produce slope triangles with special ratios. Below are the triangles you have studied so far. Note that the angles below are rounded to the nearest degree.



4.1.3 What if the angle changes?

Expanding the Trig Table



In the last few lessons, you found the slope ratios for several angles. However, so far you are limited to using the slope angles that are currently in your Trig Table. How can you find the ratios for other angles? And how are the angles related to the ratio?

Today your goal is to determine ratios for more angles and to find patterns. As you work today, keep the following questions in mind:

What happens to the slope ratio when the angle increases? Decreases?

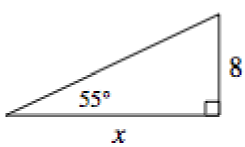
What happens to the slope ratio when the angle is 0°? 90°?

When is a slope ratio more than 1? When is it less than 1?

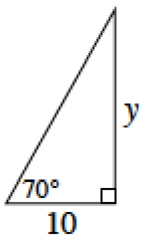
4-23. On your paper, draw a slope triangle with a slope angle of 45°.

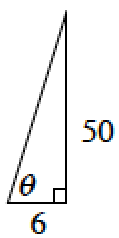
a. Now visualize what would happen to the triangle if the slope angle increased to 55°. Which would be longer? Δy or Δx? Explain your reasoning.

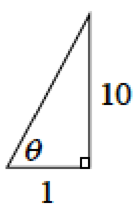
b. Using the Slope Ratios (Desmos) eTool (or the Lesson 4.1.3 Resource Page), create a triangle with a slope angle measuring 55°. Then use the resulting slope ratio to solve for x in the triangle at below. (Note: The triangle at right is not drawn to scale.)

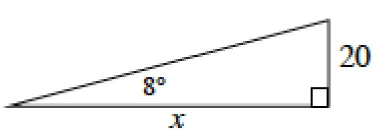


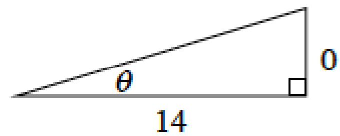
4-24. Copy each of the following triangles onto your paper. Decide whether or not the given measurements are possible. If the triangle is possible, find the value of *x*, *y*, or *ϑ*. Use the Slope Ratios (Desmos) eTool to find the appropriate slope angles or ratios needed. If technology is not available, your teacher will provide a Lesson 4.1.3 Resource Page with the needed ratios. Round angle measures to the nearest degree. If a triangle's indicated measurement is not possible, explain why.

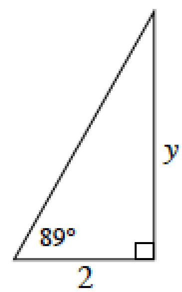
a.


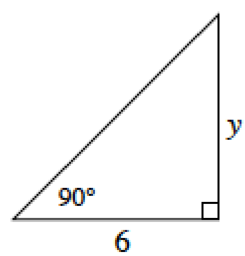
b.


c.


d.


e.


f.


g.


4-25. If you have not already, add these new slope ratios with their corresponding angles to your Trig Table Toolkit. Be sure to draw and label the triangle for each new angle. Summarize your findings—which slope triangles did not work? Do you see any patterns?

4-26. LEARNING LOG

What statements can you make about the connections between slope angle and slope ratio? In your Learning Log, write down all of your observations from this lesson. Be sure to answer the questions given at the beginning of the lesson (reprinted below). Title this entry, "Slope Angles and Slope Ratios" and include today's date.

What happens to the slope ratio when the angle increases? Decreases?

What happens to the slope ratio when the angle is 0°? 90°?

When is a slope ratio more than 1? When is it less than 1?

METHODS AND MEANINGS

MATH NOTES

Sequences

A sequence is a function in which the independent variable is a positive integer (usually called the "term number") and the dependent value is the term value. A sequence is usually written as a list of numbers.

Arithmetic Sequences

In an arithmetic sequence, the **common differenc**e between terms is constant. For example, in the arithmetic sequence 4, 7, 10, 13, …, the common difference is 3.

The equation for an arithmetic sequence is: $t(n) = mn + b$ or $a_n = mn + a_0$ where n is the term number, m is the common difference, and b or a_0 is the zeroth term. Compare these equations to a continuous linear function $f(x) = mx + b$ where m is the growth (slope) and b is the starting value (y-intercept).

For example, the arithmetic sequence 4, 7, 10, 13, … could be represented by $t(n) = 3n + 1$ or by $a_n = 3n + 1$. (Note that "4" is the first term of this sequence, so "1" is the zeroth term.)

Another way to write the equation of an arithmetic sequence is by using the first term in the equation, as in $a_n = m(n - 1) + a_1$, where is the first term. The sequence in the example could be represented by $a_n = 3(n - 4) + 13$.

You could even write an equation using any other term in the sequence. The equation using the fourth term in the example would be $a_n = 3(n - 4) + 13$.

Geometric Sequences

In a geometric sequence, the **common ratio** or **multiplier** between terms is constant. For example, in the geometric sequence 6, 18, 54, …, the multiplier is 3. In the geometric sequence 32, 8, 2, $\frac{1}{2}$, …, the common multiplier is $\frac{1}{4}$.

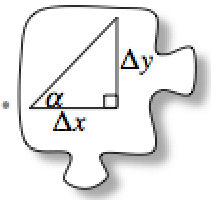
The equation for a geometric sequence is: $t(n) = ab^n$ or $a_n = a_0 \cdot b^n$ where n is the term number, b is the sequence generator (the multiplier or common ratio), and a or a_0 is the zeroth term. Compare these equations to a continuous exponential function $f(x) = ab^x$ where b is the growth (multiplier) and a is the starting value (y-intercept).

For example, the geometric sequence 6, 18, 54, … could be represented by $t(n) = 2 \cdot 3^n$ or by $a_n = 2 \cdot 3^n$.

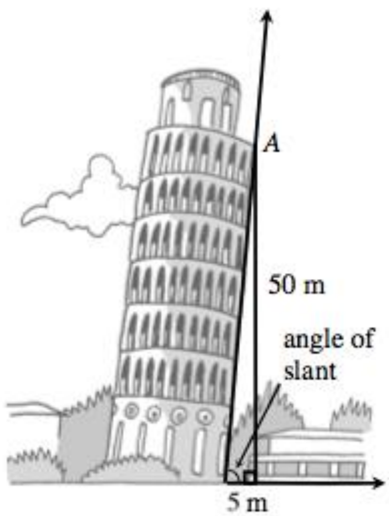
You can write a first term form of the equation for a geometric sequence as well: $a_n = a_1 \cdot b^{n-1}$. For the example, first term form would be $a_n = 6 \cdot 3^{n-1}$.

4.1.4 What about other right triangles?

The Tangent Ratio



In Lesson 4.1.2 you started a Trig Table Toolkit of angles and their related slope ratios. Unfortunately, you only have information for a few angles. How can you quickly find the ratios for other angles when a computer is not available or when an angle is not on your Trig Table? Do you have to draw each angle to get its slope ratio? Or is there another way?



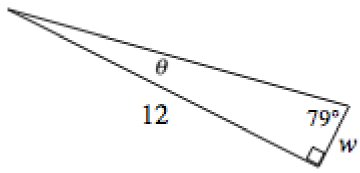
4-33. WILL IT TOPPLE?

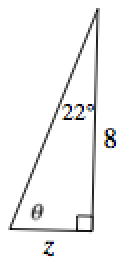
In problem 4-1, you learned that the Leaning Tower of Pisa is expected to collapse once its angle of slant is less than 83°. Currently, the top of the seventh story (point *A* in the diagram at right) is 50 meters above the ground. In addition, when a weight is dropped from point *A*, it lands 5 meters from the base of the tower, as shown in the diagram.

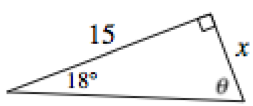
a. What is the slope ratio for the tower?

b. Use your Trig Table Toolkit to determine the angle at which the Leaning Tower of Pisa slants. Is it in immediate danger of collapse?

4-34. Solve for the variables in the triangles below. It may be helpful to first orient the triangle (by rotating your paper or by using tracing paper) so that the triangle resembles a slope triangle. Use your Trig Table for reference.

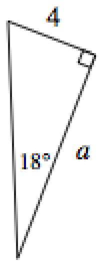
a. 

b. 

c. 

4-35. MULTIPLE METHODS

Tiana, Mae Lin, Eddie, and Amy are looking at the triangle below and trying to find the missing side length.



a. Tiana declares, "*Hey! We can rotate the triangle so that 18° looks like a slope angle, and then* $\Delta y = 4$*.*" Will her method work? If so, use her method to solve for a. If not, explain why not.

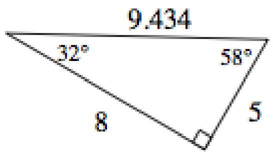
b. Mae Lin says, *"I see it differently. I can tell* $\Delta y = 4$ *without turning the triangle."* How can she tell? Explain one way she could know.

c. Eddie replies, *"What if we use 72° as our slope angle? Then* $\Delta x = 4$*."* What is he talking about? Discuss with your team and explain using pictures and words.

d. Use Eddie's observation in part (c) to confirm your answer to part (a).

4-36. USING A SCIENTIFIC CALCULATOR

Examine the triangle below.



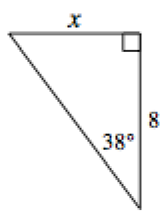
a. According to the triangle at right, what is the slope ratio for 32°? Explain how you decided to set up the ratio. Write the ratio in both fraction and decimal form.

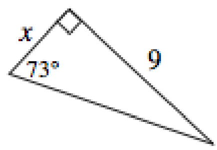
b. What is the slope ratio for the 58° angle? How do you know?

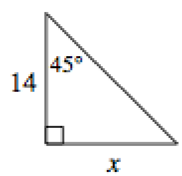
c. Scientific calculators have a button that will give the slope ratio when the slope angle is entered. In part (a), you calculated the slope ratio for 32° as 0.625. Use the "tan" button on your calculator to verify that you get approximately 0.625 when you enter 32°. Does that button give you approximately 1.600 when you enter 58°? Be ready to help your teammates find and use the button on their calculators.

d. The ratio in a right triangle that you have been studying is referred to as the **tangent ratio**. When you want to find the slope ratio of an angle, such as 32°, it is written "tan 32°." So, an equation for this triangle can be written as tan 32° = $\frac{5}{8}$. Read more about the tangent ratio in the Math Notes box for this lesson.

4-37. For each triangle below, trace the triangle on tracing paper. Label its legs Δy and Δx based on the given slope angle. Then write an equation (such as $\tan 14° = \frac{x}{5}$), use your scientific calculator to find a slope ratio for the given angle, and solve for x.

a. 

b. 

c. 

4-38. LEARNING LOG

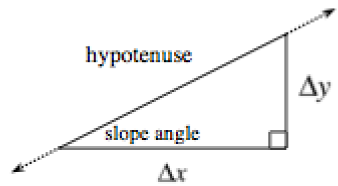
How do you set up a tangent ratio equation? How do you know which side of the triangle is Δy? How can you use your scientific calculator to find a slope ratio? Write a Learning Log entry about what you learned today. Be sure to include examples or refer to your work from today. Title this entry "The Tangent Ratio" and include today's date.

METHODS AND MEANINGS

MATH NOTES

The Tangent Ratio

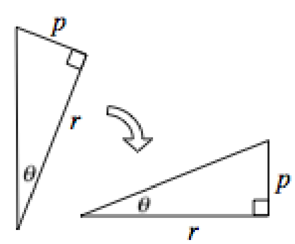
For any slope angle in a slope triangle, the ratio that compares the Δy to Δx is called the **tangent ratio**. The ratio for any angle is constant, regardless of the size of the triangle. It is written:



$$\tan(\text{slope angle}) = \frac{\Delta y}{\Delta x}$$

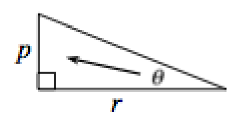
One way to identify which side is Δy and Δx is to first reorient the triangle so that it looks like a slope triangle, as shown at right.

For example, when the triangle at right is rotated, the resulting slope triangle helps to show that the tangent of θ is $\frac{p}{r}$, since θ is the slope angle, p is Δy and r is Δx. This is written:



$$\tan\ \theta = \frac{p}{r}$$

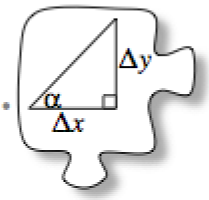
Whether the triangle is oriented as a slope triangle or not, you can identify Δy as the leg that is always opposite (across the triangle from) the angle, while Δx is the leg closest to the angle.



$$\tan\ \theta = \frac{\text{opposite leg}}{\text{adjacent leg}} = \frac{p}{r}$$

4.1.5 What if I can't measure it?

Applying the Tangent Ratio



In this section so far, you have learned how to find the legs of a right triangle using an angle. But how can you use this information? Today you and your team will use the tangent ratio to solve problems and answer questions.

4-45. STATUE OF LIBERTY

Lindy gets nosebleeds whenever she is more than 300 feet above the ground. During a class fieldtrip, her teacher asked if she wanted to climb to the top of the Statue of Liberty. Since she does not want to get a nosebleed, she decided to take some measurements to figure out the height of the torch of the statue. She found a spot directly under the torch and then measured 42 feet away and determined that the angle up to the torch was 82°. Her eyes are 5 feet above the ground. View this video about the [History of the Statue of Liberty] (YouTube).

Should she climb to the top or will she get a nosebleed? Draw a diagram that fits this situation. Justify your conclusion.

4-46. HOW TALL IS IT?

How tall is Mount Everest? How tall is the White House? Often you want to know a measurement of something you cannot easily measure with a ruler or tape measure. Today you will work with your team to measure the height of something inside your classroom or on your school's campus in order to apply your new tangent tool.

Your Task: Get a clinometer (a tool that measures a slope angle) and a meter stick (or tape measure) from your teacher. As a team, decide how you will use these tools to find the height of the object selected by your teacher. Be sure to record all measurements carefully on your Lesson 4.1.5A Resource Page and include a diagram of the situation.

Discussion Points

What should the diagram look like?

What measurements would be useful?

How can you use your tools effectively to get accurate measurements?

Independent Events

Two events are **independent** if knowing that one event occurred does not affect the probability of the other event occurring. For example, one probabilistic situation might be about a vocabulary quiz in science class today with possible outcomes {have a quiz, do not have a quiz}. Another probabilistic situation might be the outcome of this weekend's football game with the possibilities {win, lose, tie}. If you know that a quiz occurred today, it does not change the probability of the football team winning this weekend. The two events are independent.

A box contains three red chips and three black chips. If you get a red chip on the first try (and put it back in the box), the probability of getting a red chip on the second try is $\frac{3}{6}$. If you did not get a red chip on the first try, the probability of getting a red chip on the second try is still $\frac{3}{6}$. The probability of getting a red chip on the second try was not changed by knowing whether you got a red chip on the first try or not. When you return the chips to the box, the events {red on first try} and {red on second try} are independent.

However, if an event that occurred changes the probability of another event, the two events are **not independent**. Since getting up late this morning changes the probability that you will eat breakfast, these two events would not be independent.

If you get a red chip on the first try, and *do not put the first chip back in the box*, the probability of a red chip on the second try is $\frac{2}{5}$. If you did not get a red chip on the first try, the probability of getting a red chip on the second try is $\frac{3}{5}$. The probability of getting a red chip on the second try was changed by whether you got a red chip on the first try or not. When you do not replace the chips between draws, the events {red on first try} and {red on second try} are not independent.

4.2.1 How can I represent it?

Using an Area Model

In previous courses you studied probability, which is a measure of the chance that a particular event will occur. In the next few lessons you will encounter a variety of situations that require probability calculations. You will develop new probability tools to help you analyze these situations. The next two lessons focus on tools for listing *all* the possible outcomes of a probability situation, called a **sample space**.

In homework, you have practiced determining probabilities in situations where each outcome you listed had an equal probability of occurring. But what if a game is biased so that some outcomes are more likely than others? How can you represent biased games? Today you will learn a new tool to analyze more complicated situations of chance, called an area model.

4-53. IT'S IN THE GENES

Example of a thumb that can bend backwards at the joint.

Can you bend your thumb backwards at the middle joint to make an angle, like the example at right? Or does your thumb remain straight? The ability to bend your thumb back is thought to rely on a single gene.

What about your tongue? If you can roll your tongue into a "U" shape, you probably have a special gene that enables you to do this.

Assume that half of the U.S. population can bend their thumbs backwards and that half can roll their tongues. Also assume that these genes are independent (in other words, having one gene does not affect whether or not you have the other) and randomly distributed (spread out) throughout the population. Then the sample space of these genetic traits can be organized in a table like the one below.

		Bend thumb?	
		Yes $\frac{1}{2}$	No $\frac{1}{2}$
Roll Tongue?	Yes $\frac{1}{2}$		
	No $\frac{1}{2}$		

a. According to this table, what is the probability that a random person from the U.S. has both special traits? That is, what is the chance that he or she can roll his or her tongue *and* bend his or her thumb back?

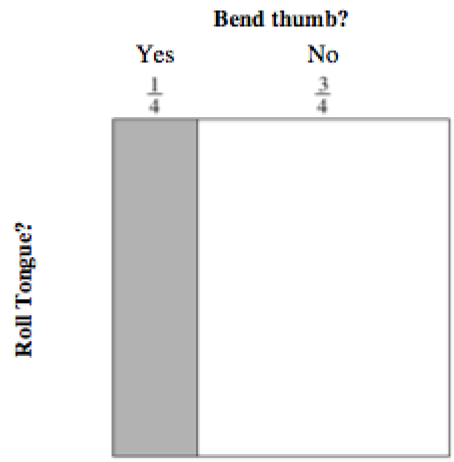
b. According to this table, what is the probability that a random person has only one of these special traits? Justify your conclusion.

c. This table is useful because every cell in the table is equally likely. Therefore, each possible outcome, such as being able to bend your thumb but not roll your tongue, has a $\frac{1}{4}$ probability.
However, this table assumes that half the population can bend their thumbs backwards, but in reality only about $\frac{1}{4}$ of the U.S. population can bend their thumbs backwards and $\frac{3}{4}$ cannot. It also turns out that a lot more (about $\frac{7}{10}$) of the population can roll their tongues. How can this table be adjusted to represent these percentages? Discuss this with your team and be prepared to share your ideas with the class.

4-54. USING AN AREA MODEL

One way to represent a sample space that has outcomes that are not equally likely is by using a **probability area model**. An area model uses a large square with an area of 1. The square is subdivided into smaller pieces to represent all possible outcomes in the sample space. The area of each outcome is the probability that the outcome will occur.

For example, if $\frac{1}{4}$ of the U.S. population can bend their thumbs back, then the column representing this ability should take only one-fourth of the square's width, as shown below.



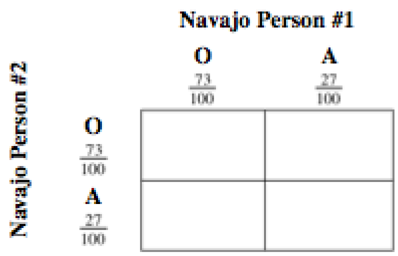
a. How should the diagram be altered to show that $\frac{7}{10}$ of the U.S. can roll their tongues? Copy this diagram on your paper and add two rows to represent this probability.

b. The relative probabilities for different outcomes are represented by the areas of the regions. For example, the portion of the probability area model representing people with both special traits is a rectangle with a width of $\frac{1}{4}$ and a height of $\frac{7}{10}$. What is the area of this rectangle? This area tells you the probability that a random person in the U.S. has both traits.

c. What is the probability that a randomly selected person can roll his or her tongue but not bend his or her thumb back? Show how you got this probability.

4-55. PROBABILITIES IN VEIN

You and your best friend may not only look different, you may also have different types of blood! For instance, members of the American Navajo population can be classified into two groups: 73% percent (73 out of 100) of the Navajo population has type "O" blood, while 27% (27 out of 100) has type "A" blood. (Blood types describe certain chemicals, called "antigens," that are found in a person's blood.)



a. Suppose you select two Navajo individuals at random. What is the probability that both individuals have type "A" blood? This time, drawing an area model that is exactly to scale would be challenging. A probability area model (like the one above) is still useful because it will still allow you to calculate the individual areas, even without drawing it to scale. Copy and complete this "generic" probability area model.

b. What is the probability that two Navajo individuals selected at random have the same blood type?

4-56. SHIPWRECKED!

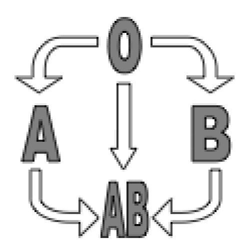
Zack and Nick (both from the U.S.) are shipwrecked on a desert island! Zack has been injured and is losing blood rapidly, and Nick is the only person around to give him a transfusion.

Unlike the Navajo you learned of in problem 4-55, most populations are classified into four blood types: O, A, B, and AB. For example, in the U.S., 45% of people have type O blood, 40% have type A, 11% have type B, and 4% have type AB (according to the American Red Cross, 2004). While there are other ways in which people's blood can differ, this problem will only take into account these four blood types.

a. Make a probability area model representing the blood types in this problem. List Nick's possible blood types along the top of the model and Zack's possible blood types along the side.

b. What is the probability that Zack and Nick have the same blood type?

c. Luckily, two people do not have to have the same blood type for the receiver of blood to survive a transfusion. Other combinations will also work, as shown in the diagram below. Assuming that their blood is compatible in other ways, a donor with type O blood can donate to receivers with type O, A, B, or AB, while a donor with type A blood can donate to a receiver with A or AB. A donor with type B blood can donate to a receiver with B or AB, and a donor with type AB blood can donate only to AB receivers.



Assuming that Nick's blood is compatible with Zack's in other ways, determine the probability that he has a type of blood that can save Zack's life!

4-57. You made a critical assumption in problem 4-56 when you made a probability area model and multiplied the probabilities.

a. Blood type is affected by genetic inheritance. What if Zack and Nick were related to each other? What if they were brothers or father and son? How could that affect the possible outcomes?

b. What has to be true in order to assume a probability area model will give an accurate theoretical probability?

METHODS AND MEANINGS

MATH NOTES

Solving a Quadratic Equation

In a previous course, you learned how to solve **quadratic equations** (equations that can be written in the form $ax^2 + bx + c = 0$). Review two methods for solving quadratic equations below.

Some quadratic equations can be solved by **factoring** and using the **Zero Product Property.** For example, because $x^2 - 3x - 10 = (x - 5)(x + 2)$, the quadratic equation $x^2 - 3x - 10 = 0$ can be rewritten as $(x - 5)(x + 2) = 0$. The Zero Product Property states that *if* $ab = 0$, then $a = 0$ or $b = 0$. So, if $(x - 5)(x + 2) = 0$, then $x - 5 = 0$ or $x + 2 = 0$. Therefore, $x = 5$ or $x = -2$.

Another method for solving quadratic equations is the **Quadratic Formula**. This method is particularly helpful for solving quadratic equations that are difficult or impossible to factor. Before using the Quadratic Formula, the quadratic equation you want to solve must be in standard form, that is, written as $ax^2 + bx + c = 0$.

In this form, a is the coefficient of the x^2 term, b is the coefficient of the x term, and c is the constant term. The Quadratic Formula states:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

This formula gives two possible answers due to the "±" symbol. This symbol (read as "plus or minus") is shorthand notation that tells us to calculate the formula twice: once using addition and once using subtraction. Therefore, every Quadratic Formula problem must be simplified twice to give:

$$x = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \quad \text{or} \quad x = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

To solve $x^2 - 3x - 10 = 0$ using the Quadratic Formula, substitute $a = 1$, $b = -3$, and $c = -10$ into the formula, as shown below.

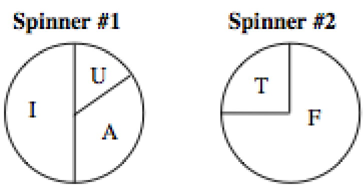
$$x = \frac{-(-3) \pm \sqrt{(-3)^2 - 4(1)(-10)}}{2(1)} \rightarrow \frac{3 \pm \sqrt{49}}{2} \rightarrow \frac{3 \pm 7}{2} \rightarrow x = 5 \text{ or } x = -2$$

4.2.2 How can I represent it?

Using a Tree Diagram

In Lesson 4.2.1, you used a probability area model to represent probability situations where some outcomes were more likely than others. Today you will consider how to represent these types of situations using tree diagrams.

4-64. Your teacher challenges you to a spinner game. You spin the two spinners with the probabilities listed below. The first letter comes from Spinner #1 and the second letter from Spinner #2. If the letters can form a two-letter English word, you win. Otherwise, your teacher wins. Test your ideas with the *4-64 Student eTool* (CPM) spinner game.



$P(I) = \frac{1}{2}$
$P(U) = \frac{1}{6}$
$P(A) = \frac{1}{3}$

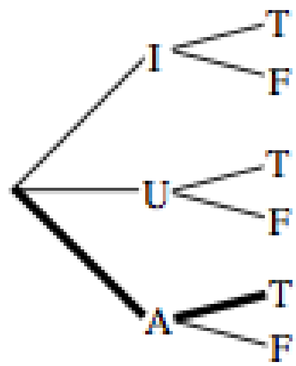
$P(T) = \frac{1}{4}$
$P(F) = \frac{3}{4}$

a. Are the outcomes for spinner #2 independent of the outcomes for spinner #1?

b. Make a probability area model of the sample space, and find the probability that you will win this game.

c. Is this game fair? If you played the game 100 times, who do you think would win more often, you or your teacher? Can you be sure this will happen?

4-65. Sinclair wonders how to model the spinner game in problem 4-64 using a tree diagram. He draws the tree diagram below.



a. Sabrina says, *"That can't be right. This diagram makes it look like all the words are equally likely."* What is Sabrina talking about? Why is this tree diagram misleading?

b. To make the tree diagram reflect the true probabilities in this game, Sabrina writes numbers on each branch showing the probability that the letter will occur. So she writes a "$\frac{1}{3}$" on the branch for "A," a "$\frac{1}{4}$" on the branch for each "T," etc. Following Sabrina's method, label the tree diagram with probabilities on each branch.

c. According to the probability area model that you made in problem 4-64, what is the probability that you will spin the word "AT"? Now examine the bolded branch on the tree diagram shown above. How could the numbers you have written on the tree diagram be used to find the probability of spinning "AT"?

d. Does this method work for the other combinations of letters? Similarly calculate the probabilities for each of the paths of the tree diagram. At the end of each branch, write its probability. (For example, write "$\frac{1}{12}$" at the end of the "AT" branch.) Do your answers match those from problem 4-64?

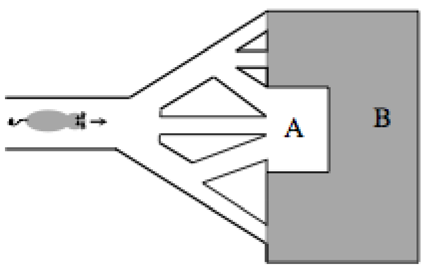
e. Find all the branches with letter combinations that make words. Use the numbers written at the end of each branch to compute the total probability that you will spin a word. Does this probability match the probability you found with your area model?

4-66. THE RAT RACE

Ryan has a pet rat Romeo that he boasts is the smartest rat in the county. Sammy overheard Ryan at the county fair claiming that Romeo could learn to run a particular maze and find the cheese at the end.

"I don't think Romeois that smart!" Sammy declares, *"I think the rat just chooses a random path through the maze."*

Ryan has built a maze with the floor plan shown below. In addition, he has placed some cheese in an airtight container (so Romeo can't smell the cheese!) in room A.

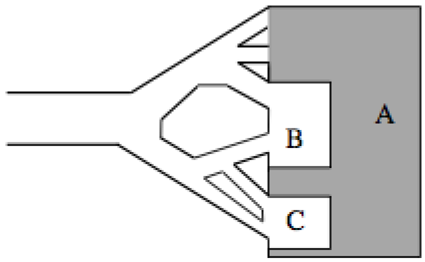


a. Suppose that every time Romeo reaches a split in the maze, he is equally likely to choose any of the paths in front of him. Choose a method and calculate the probability that Romeo will end up in each room. In a sentence or two, explain why you chose the method you did.

b. If the rat moves through the maze randomly, how many out of 100 attempts would you expect Romeo to end up in room A? How many times would you expect him to end up in room B? Explain.

c. After 100 attempts, Romeo has found the cheese 66 times. Ryan says, "*See how smart Romeo is? He clearly learned something and got better at the maze as he went along*." Sammy is not so sure.
Do you think Romeo learned and improved his ability to return to the same room over time? Or could he just have been moving randomly? Discuss this question with your team. Then, write an argument that would convince Ryan or Sammy.

4-67. Always skeptical, Sammy says, "*If Romeo really can learn, he ought to be able to figure out how to run this new maze I've designed.*" Examine Sammy's maze below.



a. To give Romeo the best chance of finding the cheese, in which room should the cheese be placed? Choose a method, show all steps in your solution process, and justify your answer.

b. If the cheese is in room C and Romeo finds the cheese 6 times out of every 10 tries, does he seem to be learning? Explain your conclusion.

4-68. LEARNING LOG

Make an entry in your Learning Log describing the various ways of representing complete sample spaces. For each method, indicate how you compute probabilities using the method. Which method seems easiest to use so far? Label this entry "Creating Sample Spaces" and include today's date. Set this Learning Log aside in a safe place. You will need it in the next lesson.

4.2.3 What model should I use?

Probability Models

In this lesson you will review ideas of probability as you use systematic lists, tree diagrams, and area models to account for all of the elements in a sample space, account for equally likely outcomes, and identify events. You will find that certain tools may work better for particular situations. In one problem a tree diagram or list might be most efficient, while in another problem an area model may be the best choice. As you work with your team, keep the following questions in mind:

What are the possible outcomes?
Are the outcomes equally likely?
Will a tree diagram, list, or area model help?
What is the probability for this event?

4-75. ROCK, PAPER, SCISSORS

Your team will play a variation of "Rock, Paper, Scissors" (sometimes called "Rochambeau") and record points. You will need to work in a team of four. Have one person act as recorder while the other three play the game.

a. List the names of the people in your team alphabetically. The first person on the list is Player A, the next is Player B, the third is Player C, and the fourth is the recorder. Write down who has each role.

b. Without playing the game, discuss with your team which player you think will receive the most points by the end of the game. Assign points as follows:

   a. Player A gets a point each time all three players match.

   b. Player B gets a point each time two of the three players match.

   c. Player C gets a point each time none of the players match.

c. Now play "Rock, Paper, Scissors" with your team at least 20 times. The recorder should record the winner for each round. Does this game seem fair?

d. Calculate the theoretical probability for each outcome (Player A, Player B, or Player C winning). Discuss this with your team and be prepared to share your results with the class.

e. Devise a plan to make this game fair.

4-76. There is a new game at the school fair called "Pick a Tile," in which the player reaches into two bags and chooses one square tile and one circular tile. The bag with squares contains three yellow, one blue, and two red squares. The bag with circles has one yellow and two red circles. In order to win the game (and a large stuffed animal), a player must choose one blue square and one red circle. Explore this game using the *4-76 Student eTool* (CPM).

Since it costs $2 to play the game, Marty and Gerri decided to calculate the probability of winning before deciding whether to play.

Gerri suggested making a systematic list of all the possible color combinations in the sample space, listing squares first then circles:

$$RY \quad BY \quad YY$$

$$RR \quad BR \quad YR$$

*"So,"*says Gerri, *"the answer is* $\frac{1}{6}$ *."*

"That doesn't seem quite right," says Marty. *"There are more yellow squares than blue ones. I don't think the chance of getting a yellow square and a red circle should be the same as getting a blue square and a red circle."*

a. Make a tree diagram for this situation. Remember to take into account the duplicate tiles in the bags.

b. Find the probability of a player choosing the winning blue square-red circle combination.

c. Should Gerri and Marty play this game? Would you? Why or why not?

4-77. Now draw a probability area model for the “Pick a Tile” game in problem 4-76.

a. Use the probability area model to calculate the probability of each possible color combination of a square and a circular tile.

b. Explain to Marty and Gerri why the probability area model is called an *area* model.

c. Discuss which model you preferred using to solve the “Pick a Tile” problem with your team. What are your reasons for your preference?

d. Could you have used the area model for the “Rock, Paper, Scissors” problem? Explain why or why not.

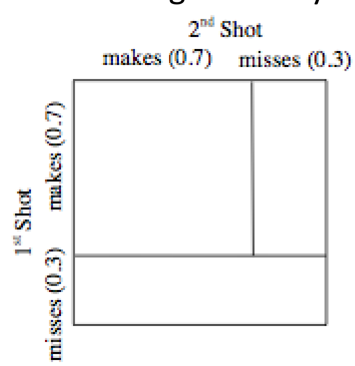
4-78. BASKETBALL: Shooting One-and-One Free Throws

Rimshot McGee has a 70% free throw average. The opposing team is ahead by one point. Rimshot is at the foul line in a one-and-one situation with just seconds left in the game. (A one-and-one situation means that the player shoots a free throw. If they make the shot, they are allowed to shoot another. If they miss the first shot, they get no second shot. Each shot made is worth one point.)

a. First, take a guess. What do you think is the most likely outcome for Rimshot: zero points, one point, or two points?

b. Draw a tree diagram to represent this situation.

c. Jeremy is working on the problem with Jenna and he remembers that area models are sometimes useful for solving problems related to probability. They set up the probability area model below. Discuss this model with your team. Which part of the model represents Rimshot getting one point? How can you use the model to help calculate the probability that Rimshot will get exactly one point?



d. Use either your tree diagram or the area model to help you calculate the probabilities that Rimshot will get either 0 or 2 points. What is the most likely of the three outcomes?

4-79. With your team, examine the probability area model from problem 4-78.

a. What are the dimensions of the large rectangle? Explain why these dimensions make sense.

b. What is the total area of the model? Express the area as a product of the dimensions and as a sum of the parts.

c. What events are represented by the entire area model?

4-80. This Learning Log extends the entry that you made in problem 4-68. In that entry you described the various ways of representing complete sample spaces and showed how to use each method to find probabilities.

Expand upon your entry. Are there any conditions under which certain methods to represent the sample space can or cannot be used? Which methods seem most versatile? Why? Title this entry "Conditions For Using Probability Methods" and include today's date.

METHODS AND MEANINGS

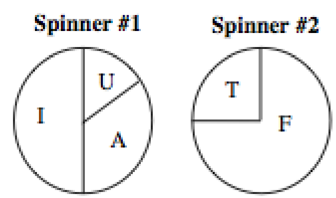
MATH NOTES

Probability Models

When all the possible outcomes of a probabilistic event are *equally likely*, you can calculate probabilities as follows:

$$\text{Theoretical probability} = \frac{\text{number of successful outcomes}}{\text{total number of possible outcomes}}$$

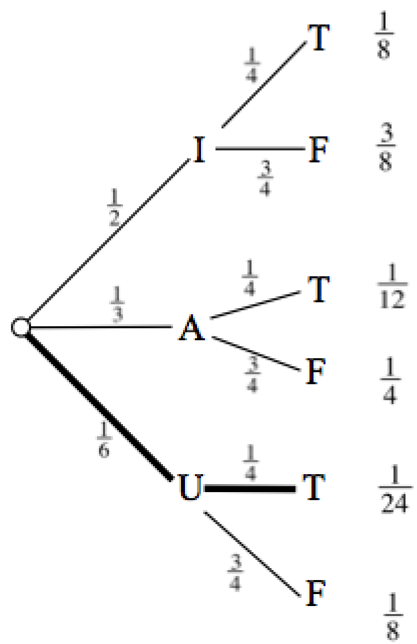
But suppose you spin the two spinners shown at right. These outcomes are not all equally likely so another model is needed to calculate probabilities of outcomes.



A **probability area model** is practical if there are exactly two probabilistic situations and they are independent. The outcomes of one probabilistic situation are across the top of the table, and the outcomes of the other are on the left. The smaller rectangles are the sample space. Then the probability for an outcome is the area of the rectangle. For example, the probability of spinning "UT" is $\frac{1}{6} \cdot \frac{1}{4} = \frac{1}{24}$. Notice that the area (the probability) of the large overall square is 1.

Spinner #2 \ Spinner #1	I $\left(\frac{1}{2}\right)$	A $\left(\frac{1}{3}\right)$	U $\left(\frac{1}{6}\right)$
T $\left(\frac{1}{4}\right)$	IT $\left(\frac{1}{8}\right)$	AT $\left(\frac{1}{12}\right)$	UT $\left(\frac{1}{24}\right)$
F $\left(\frac{3}{4}\right)$	IF $\left(\frac{3}{8}\right)$	AF $\left(\frac{1}{4}\right)$	UF $\left(\frac{1}{8}\right)$

A **tree diagram** can be used even if there are more than two probabilistic situations, and the events can be independent or not. In this model, the ends of the branches indicate outcomes of probabilistic situations, and the branches show the probability of each event. For example, in the tree diagram at right the first branching point represents Spinner #1 with outcomes “I” “A” or “U”. The numbers on the branch represent the probability that a letter occurs.



The numbers at the far right of the table represent the probabilities of various outcomes. For example, the probability of spinning “U” and “T” can be found at the end of the bold branch of the tree. This probability, $\frac{1}{24}$, can be found by multiplying the fractions that appear on the bold branches.

4.2.4 What if both events happen?

Unions, Intersections, and Complements

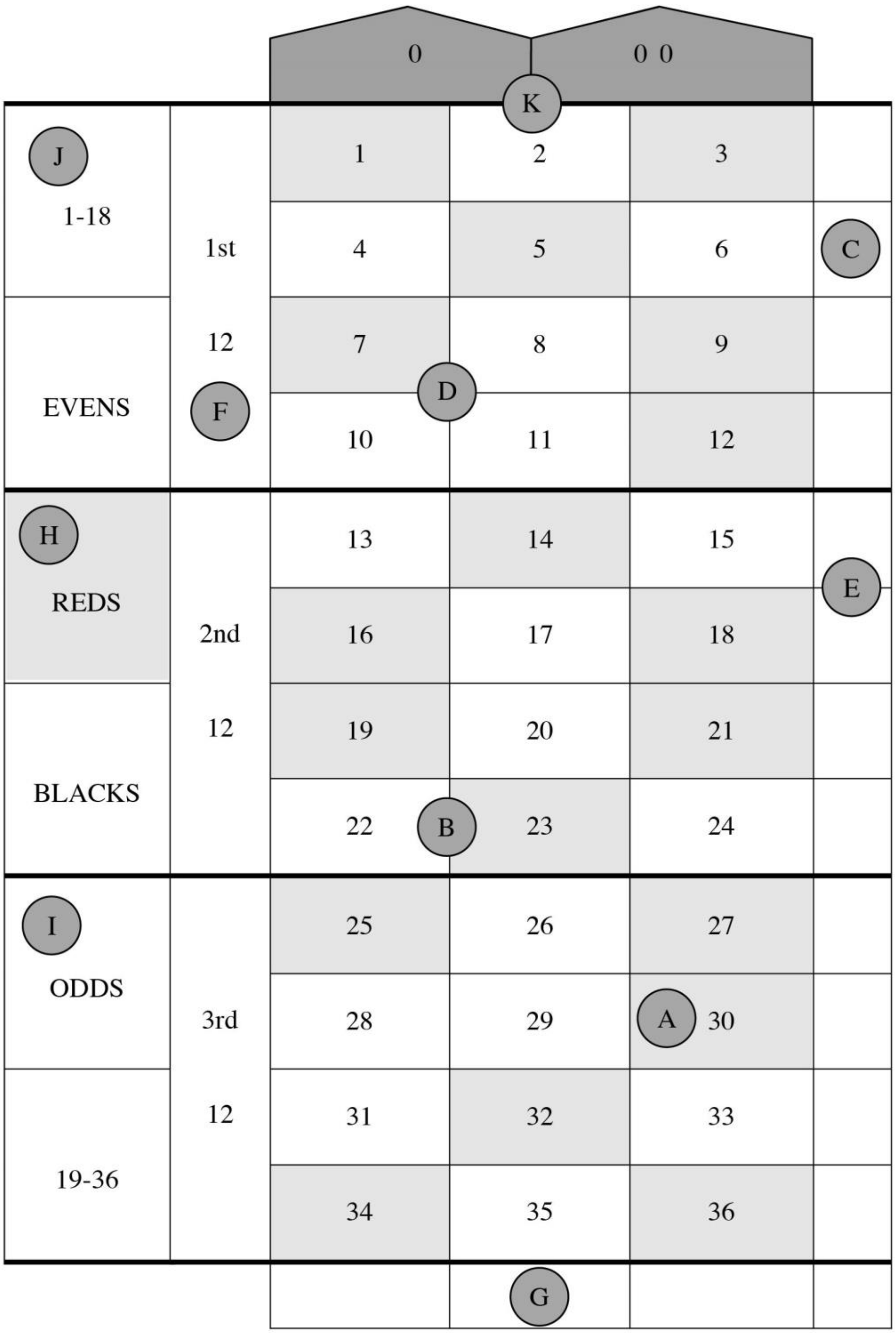
In the mid 1600's, a French nobleman, the Chevalier de Mere, was wondering why he was losing money on a bet that he thought was a sure winner. He asked the mathematician Blaise Pascal, who consulted with another mathematician, Pierre de Fermat. Together they solved the problem, and this work provided a beginning for the development of probability theory. Since argument over the analysis of a dice game provided a basis for the study of this important area of mathematics, casino games are a reasonable place to continue to investigate and clarify the ideas and language of probability.

As one of the simplest casino games to analyze, roulette is a good place to start. In American roulette the bettor places a bet, the croupier (game manager) spins the wheel and drops the ball and then everyone waits for the ball to land in one of 38 slots. The 38 slots on the wheel are numbered 00, 0, 1, 2, 3, …, 36. Eighteen of the numbers are red and eighteen are black; 0 and 00 are green. (In French roulette, also known as Monte Carlo, there is no 00, so there are only 37 slots on the wheel.)

Before the ball is dropped, players place their chips on the roulette layout, shown below. Bets can be placed on:

- A single number;
- Two numbers by placing the chip on the line between them;
- Three numbers by placing a chip on the line at the edge of a row of three;
- Four numbers by placing the chip where the four corners meet;
- Five numbers (0, 00, 1, 2, 3);
- Six numbers by placing the bet at an intersection at the edge;
- A column, the 1^{st} twelve, 2^{nd} twelve, or 3^{rd} twelve;
- Even numbers;
- Odd numbers;
- 1-18;
- 19-36;
- Red numbers; or,
- Black numbers

- Note: The lightly shaded numbers, 1, 3, 5, 7, 9, 12, 14, 16, 18, 19, 21, 23, 25, 27, 30, 32, 34, and 36 are red.


0
0 0
K
J
1-18
EVENS
H
REDS
BLACKS
I
ODDS
19-36
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F
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G

4-87. Obtain a Lesson 4.2.4A Resource Page from your teacher. On the resource page, the "chips" A through K represent possible bets that could be made. Use the following *4-87 Student eTool* (CPM) to generate numbers on a roulette wheel.

a. What is the sample space for one spin of the roulette wheel?

b. Are the outcomes equally likely?

c. A subset (smaller set) of outcomes from the sample space is called an event. For example, chip A represents the event {30}, and chip B represents the event {22, 23}. Make a list of events and their probabilities for Chips A-K.

4-88. Some roulette players like to make two (or more) bets at the same time. A bettor places a chip on the event {7, 8, 10, 11} and then another chip on the event {10, 11, 12, 13, 14, 15}. What numbers will allow the bettor to win both bets? Next find the bettor's chances of winning the bet of the first chip *and* winning the bet of the second chip on a single drop of the roulette ball. This is called the probability of the ball landing on a number that is in the **intersection of two events**.

4-89. When placing two different bets, most players are just hoping that they will win on one *or* the other of the two events. The player is betting on the **union** of two events.

a. Calculate the probability of winning either the bet on the event {7, 8, 10, 11} *or* the bet on the event {10, 11, 12, 13, 14, 15}. Think about the set of outcomes that will allow the bettor to win either of the bets. This set of outcomes is the union of the two events.

b. Calculate the probability of the union of {numbers in first column} and {"2^{nd} 12" numbers 13 through 24}.

c. One bettor's chip is on the event {13, 14, 15, 16, 17, 18} and another on {Reds}. What is the probability of the union of these events?

d. Explain your method for finding the probabilities in parts (a) through (c).

4-90. Viola described the following method for finding the probability for part (a) of problem 4-89:

"When I looked at the probability of either of two events, I knew that would include all of the numbers in both events, but sometimes some numbers might be counted twice. So, instead of just counting up all of the outcomes, I added the two probabilities together and then subtracted the probability of the overlapping events or numbers. So it's just $\frac{4}{38}+\frac{6}{38}-\frac{2}{38}=\frac{8}{38}$*."*

Does Viola's method always work? Why or why not? Is this the method that you used to do problem 4-89? If not show how to use Viola's method on one of the other parts of problem 4-89.

4-91. Viola's method of "adding the two probabilities and subtracting the probability of the overlapping event" is called the **Addition Rule** and can be written:

$$P(A \textit{ or } B) = P(A) + P(B) - P(A \textit{ and } B)$$

You have already seen that any event that includes event A *or* event B can be called a union, and is said "A union B." The event where both events A *and* B occur together is called an intersection. So the Addition Rule can also be written:

$$P(A \text{ union } B) = P(A) + P(B) - P(A \text{ intersection } B)$$

Use these ideas to do the following: A player places a chip on the event {1-18} and another on the event {Reds}. Consider the event {1-18} as event "A", and the event {Reds} as event "B." Clearly show two different ways to figure out the probability of the player winning one of the two bets.

4-92. On the Lesson 4.2.4A Resource Page, consider a player who puts a chip on both events "G" and "I."

a. How does the event {G *or* I} differ from the event {G *and* I}?

b. List the set of outcomes for the intersection of events G and I, and the set of outcomes for the union of events G or I.

c. Is the player who puts a chip on both G and I betting on the "or" or the "and?" Use both the counting method and the Addition Rule to find the probability that this player will win.

4-93. Wyatt places a bet on event G.

a. What is the probability that he will lose?

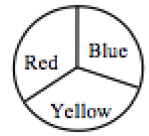
b. How did you calculate the probability of {not event G}?

c. Show another method for calculating the probability of the bettor losing on event G.

4-94. Sometimes it is easier to figure out the probability that something will *not* happen than the probability that it *will*. When finding the probability that something will not happen, you are looking at the **complement** of an event. The complement is the set of all outcomes in the sample space that are not included in the event.

Show two ways to solve the problem below, then decide which way you prefer and explain why. Test your ideas using the *4-94 Student eTool*.

a. Crystal is spinning the spinner at right and claims she has a good chance of having the spinner land on red at least once in three tries. What is the probability that the spinner will land on red at least once in three tries?



b. If the probability of an event A is represented symbolically as P(A), how can you symbolically represent the probability of the complement of event A?

METHODS AND MEANINGS

MATH NOTES

Probability Models

When all the possible outcomes of a probabilistic event have the same probability, probabilities can be calculated by listing the possible outcomes in a **systematic list**. However, when some outcomes are more probable than others, a more sophisticated model is required to calculate probabilities.

A smaller set of outcomes from a sample space is called an **event**. For example, if you draw one card from a standard deck of 52 cards, the sample space would be {A♠, A♣, A♥, A♦, 2♠, 2♣, 2♥, 2♦, … , K♠, K♣, K♥, K♦}. An event might be {drawing a spade}, which would be set {A♠, 2♠, 3♠, … , Q♠, K♠}. The event {drawing a face card} is the set {J♠, J♣, J♥, J♦, Q♠, Q♣, Q♥, Q♦, K♠, K♣, K♥, K♦}.

The **complement** of an event is all the outcomes in the sample space that are not in the original event. For example, the complement of {drawing a spade} would be all the hearts, diamonds, and clubs, represented as the complement of {drawing a spade} = {A♥, 2♥, 3♥, …, Q♥, K♥, A♦, 2♦, 3♦, … , Q♦, K♦, A♣, 2♣, 3♣, … , Q♣, K♣}.

The **intersection of two events** is the event in which *both* the first event *and* the second event occur. The intersection of the events {drawing a spade} and {face card} would be {J♠, Q♠, K♠} because these three cards are in both the event {drawing a spade} *and* the event {face card}.

The **union** of two events is the event in which the first event *or* the second event (or both) occur. The union of the events {drawing a spade} or a {face card} is {A♠, 2♠, 3♠, 4♠, 5♠, 6♠, 7♠, 8♠, 9♠, 10♠, J♠, Q♠, K♠, J♣, J♥, J♦, Q♣, Q♥, Q♦, K♣, K♥, K♦}. This event has 22 outcomes.

The probability of *equally likely* events can be found by: $\dfrac{\text{number of successful outcomes}}{\text{total number of possible outcomes}}$

The probability of {drawing a spade} or {drawing a face card} is $\frac{22}{52}$ *because there are 22 cards in the union and 52 cards in the sample space.*

Alternatively, the probability of the union of two events can be found by using the **Addition Rule**:

P(A or B) = P(A) + P(B) – P(A and B)

or

P(A union B) = P(A) + P(B) – P(A intersection B)

If you let event A be {drawing a spade} and event B be {drawing a face card},

P(A) = P(spade) = , P(B) = P(face card) = $\frac{12}{52}$,

P(A and B) = P(spade and face card) = $\frac{3}{52}$.

Then, the probability of drawing a spade *or* a face card is:

P(A or B) = P(A) + P(B) – P(A and B) = $\frac{13}{52}+\frac{12}{52}-\frac{3}{52}=\frac{22}{52}$.

4.2.5 How much can I expect to win?

Expected Value

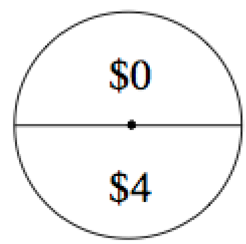
Different cultures have developed creative forms of games of chance. For example, native Hawaiians play a game called Konane, which uses markers and a board and is similar to checkers. Native Americans play a game called To-pe-di, in which tossed sticks determine how many points a player receives.

When designing a game of chance, attention must be given to make sure the game is fair. If the game is not fair, or if there is not a reasonable chance that someone can win, no one will play the game. In addition, if the game has prizes involved, care needs to be taken so that prizes will be distributed based on their availability. In other words, if you only want to give away one grand prize, you want to make sure the game is not set up so that 10 people win the grand prize!

Today your team will analyze different games to learn about expected value, which helps to predict the result of a game of chance.

4-101. TAKE A SPIN

Consider the following game: After you spin the wheel below, you win the amount spun. Explore using the *4-101 Student eTool* (CPM).

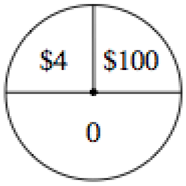


a. If you play the game 10 times, how much money would you expect to win? What if you played the game 30 times? 100 times? Explain your process.

b. What if you played the game *n* times? Write an equation for how much money someone can expect to win after playing the game *n* times.

c. If you were to play only once, what would you expect to earn according to your equation in part (b)? Is it actually possible to win that amount? Explain why or why not.

4-102. What if the spinner looks like the one below instead? Explore using the *4-102 Student eTool* (CPM).

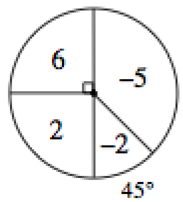


a. If you win the amount that comes up on each spin, how much would you expect to win after 4 spins? What about after 100 spins?

b. Find this spinner's **expected value**. That is, what is the expected amount you will win for each spin? Be ready to justify your answer.

c. Gustavo describes his thinking this way: *"Half the time, I'll earn nothing. One-fourth the time, I'll earn $4 and the other one-fourth of the time I'll earn $100. So, for one spin, I can expect to win* $\frac{1}{2}$ (0) + $\frac{1}{4}$ ($4) + $\frac{1}{4}$ ($100)." Calculate Gustavo's expression. Does his result match your result from part (b)?

4-103. Jesse has created the spinner at right. This time, if you land on a positive number, you win that amount of money. However, if you land on a negative number, you lose that amount of money! Want to try it? Explore using the *4-103 Student eTool* (CPM).



a. Before analyzing the spinner, predict whether a person would win money or lose money after many spins.

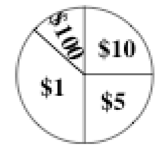
b. Now calculate the actual expected value. How does the result compare to your estimate from part (a)?

c. What would the expected value be if this spinner were fair? Discuss this with your team. What does it mean for a spinner to be fair?

d. How could you change the spinner to make it fair? Draw your new spinner and show why it is fair.

4-104. DOUBLE SPIN

"Double Spin" is a new game. The player gets to spin a spinner twice, but wins only if the same amount comes up both times. The \$100 sector is $\frac{1}{8}$ of the circle.



a. Use an area model or tree diagram to show the sample space and probability of each outcome of two spins and then answer the following questions.

b. What is the expected value when playing this game? That is, what is the average amount of money the carnival should expect to pay to players each turn over a long period of time?

c. If it costs \$3.00 for you to play this game, should you expect to break even in the long run?

d. Is this game fair?

4-105. Basketball: Shooting One-and-One Free Throws Revisited

Recall the One-and-One situation from problem 4-78. In this problem, Dunkin' Delilah Jones has a 60% free throw average.

a. Use an appropriate model to represent the sample space and then find what would be the most likely result when she shoots a one-and-one.

b. Is it more likely that Delilah would make no points or that she would score some points? Explain.

c. On average, how many points would you expect Dunkin' Delilah to make in a one and one free throw situation? That is, what is the expected value?

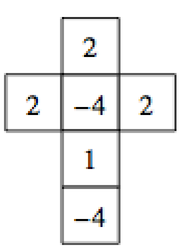
d. Repeat part (a) for at least three other possible free throw percentages, making a note of the most likely outcome for each one.

e. Is there a free throw percentage that would make two points and zero points equally likely outcomes? If so, find this percentage.

f. If you did not already do so, draw an area model or tree diagram for part (e) using x as the percentage and write an equation to represent the problem. Write the solution to the equation in simplest radical form.

4-106. Janine's teacher has presented her with an opportunity to raise her grade: She can roll a special die and possibly gain points. If a positive number is rolled, Janine gains the number of points indicated on the die. However, if a negative roll occurs, then Janine loses that many points.

Janine does not know what to do! The die, formed when the net below is folded, offers four sides that will increase her number of points and only two sides that will decrease her grade. She needs your help to determine if this die is fair.



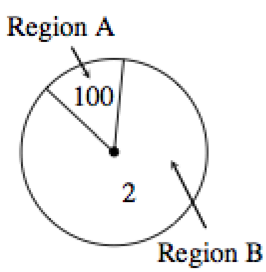
a. What are the qualities of a fair game? How can you tell if a game is fair? Discuss this with your team and be ready to share your ideas with the class.

b. What is the expected value of one roll of this die? Show how you got your answer. Is this die fair?

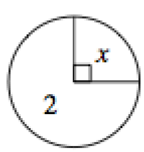
c. Change only one side of the die in order to make the expected value 0.

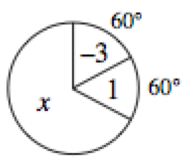
d. What does it mean if a die or spinner has an expected value of 0?

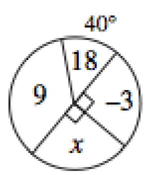
4-107. Examine the spinner below. If the central angle of Region A is 7°, find the expected value of one spin two different ways. Be ready to share your methods with the class.



4-108. Now reverse the process. For each spinner below, find x so that the expected value of the spinner is 3. Be prepared to explain your method to the class.

a. 

b. 

c. 

4-109. Revisit your work from part (c) of problem 4-108.

a. To solve for *x*, Julia wrote the equation:
$\frac{140}{360}(9)+\frac{40}{360}(18)+\frac{90}{360}(-3)+\frac{90}{360}x=3$
Explain how her equation works.

b. She is not sure how to solve her equation. She would like to rewrite the equation so that it does not have any fractions. What could she do to both sides of the equation to eliminate the fractions? Rewrite her equation and solve for *x*.

c. If you have not done so already, write an equation and solve for *x* for parts (a) and (b) of problem 4-108. Did your answers match those you found in problem 4-108?